Vježbe iz

UAM-a I

(1-19)

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Termodinamika

I zakon $\quad dU = d'Q - d'A \qquad d'A = \sum_i R_i\, dr_i = PdV - EdP - \mathcal{H}dM + \dots$

$$d'Q = dU + PdV - EdP - \mathcal{H}dM \qquad (1)$$

C topl. kapacitet → način mjerenja $C_V, C_P, \dots C_{\mathcal{H}}$

$$C_i = \left(\frac{d'Q}{\partial T}\right)_i \qquad (2)$$

Probl.

1. Izračunati razlik topl. kapacitet $C_{\mathcal{H}} - C_M$

uz jbu stanja $\quad M = a\dfrac{\mathcal{H}}{T}$ Curie (3)

$$(1) \Rightarrow d'Q = \left(\frac{\partial U}{\partial T}\right)_M dT + \left(\frac{\partial U}{\partial M}\right)_T dM - \mathcal{H}dM \qquad \text{jer } U(T,M)$$

$$= \left(\frac{\partial U}{\partial T}\right)_M dT + \left[\left(\frac{\partial U}{\partial M}\right)_T - \mathcal{H}\right] dM$$

$$C_M = \left(\frac{d'Q}{\partial T}\right)_M = \left(\frac{\partial U}{\partial T}\right)_M$$

$$C_{\mathcal{H}} = \left(\frac{d'Q}{\partial T}\right)_{\mathcal{H}} = C_M + \left[\left(\frac{\partial U}{\partial M}\right)_T - \mathcal{H}\right]\left(\frac{\partial M}{\partial T}\right)_{\mathcal{H}}$$

Izt. d ⇒ $d'Q = T\,dS$ tot. dif.

$$dS = \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_M dT + \frac{1}{T}\left[\left(\frac{\partial U}{\partial M}\right)_T - \mathcal{H}\right] dM$$

$$\Rightarrow \left[\frac{\partial}{\partial M}\frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_M\right]_T = \left[\frac{\partial}{\partial T}\frac{1}{T}\left(\left(\frac{\partial U}{\partial M}\right)_T - \mathcal{H}\right)\right]_M$$

$$-\frac{1}{T^2}\left(\frac{\partial U}{\partial T}\right)_M \underbrace{\left(\frac{\partial T}{\partial M}\right)_T}_{0} + \cancel{\frac{1}{T}\left[\frac{\partial}{\partial M}\left(\frac{\partial U}{\partial T}\right)_M\right]_T} = \cancel{\frac{1}{T}\left[\frac{\partial}{\partial T}\left(\frac{\partial U}{\partial M}\right)_T\right]_M} - \frac{1}{T}\left(\frac{\partial \mathcal{H}}{\partial T}\right)_M - \frac{1}{T^2}\left[\left(\frac{\partial U}{\partial M}\right)_T - \mathcal{H}\right]\left(\frac{\partial T}{\partial T}\right)_M$$

$$\Rightarrow \left(\frac{\partial U}{\partial M}\right)_T = -T\left(\frac{\partial \mathcal{H}}{\partial T}\right)_M + \mathcal{H} \qquad \text{jbn. energije}$$

$$C_{\mathcal{H}} - C_M = \left[\left(\frac{\partial U}{\partial M}\right)_T - \mathcal{H} \right] \left(\frac{\partial M}{\partial T}\right)_{\mathcal{H}} = -T \left(\frac{\partial \mathcal{H}}{\partial T}\right)_M \left(\frac{\partial M}{\partial T}\right)_{\mathcal{H}}$$

$M = a \frac{\mathcal{H}}{T}$ $\quad \left(\frac{\partial \mathcal{H}}{\partial T}\right)_M = \frac{M}{a} \quad \left(\frac{\partial M}{\partial T}\right)_{\mathcal{H}} = -\frac{a\mathcal{H}}{T^2}$

$$\Rightarrow \underline{C_{\mathcal{H}} - C_M = \frac{M \cdot \mathcal{H}}{T}}$$

analogno: $\mathcal{P} = b \frac{E}{T} \quad \Rightarrow C_E - C_{\mathcal{P}} = \frac{\mathcal{P} \cdot E}{T}$

2. Ovisnost entropije 1-A idealn. plina o volum. i temp. $S(V,T)$

$TdS = dU + PdV \qquad PV = nRT \qquad U = N \cdot 3 \cdot \frac{kT}{2} = n \frac{3}{2} RT$

$U = \frac{3}{2} nRT \qquad \left(\frac{\partial U}{\partial V}\right)_T = 0$

$$TdS = \frac{3}{2} nR\,dT + \frac{nRT}{V} dV$$

$$dS = \frac{3}{2} nR \frac{dT}{T} + \frac{nR}{V} dV$$

$$\Rightarrow S = \frac{3}{2} nR \ln T + nR \ln V + S_0$$

II.

1. Pokazati da za sistem s unut. energijom $U = \underset{\text{const.}}{A} V T^n \quad n \neq 1$ vrijedi $G = 0$

$G = U + PV - TS$

2. Magnetizacija idealnog paramagneta promijeni se sa $M_1 \to M_2$. Kolika je promjena entropije, ako su tokom procesa T i V bili konstantni. Jba. stanja $\to$ Curie $M = a \frac{\mathcal{H}}{T}$ $\Rightarrow$ $\left.\begin{array}{l} dU = 0 \\ \& \left(\frac{\partial U}{\partial M}\right)_T = 0 \end{array}\right\}$

$\Rightarrow TdS = -\mathcal{H}dM$

$dS = -\frac{\mathcal{H}}{T} dM \qquad \cancel{S_2 - S_1 = -\frac{\mathcal{H}}{T}(M_2 - M_1)}$

3. Pokazati da iz uvjeta

$$\left(\frac{\partial H}{\partial P}\right)_T = 0 \quad \& \quad \left(\frac{\partial U}{\partial V}\right)_T = 0$$

da za sistem vrijedi jba. idealnog plina

$$TdS = dU + PdV$$

$$H = U + PV$$

$$dH = dU + PdV + VdP$$

$$TdS = dH - VdP$$

$H(T,P)$

$$dS = \frac{1}{T}dH - \frac{V}{T}dP = \frac{1}{T}\left(\frac{\partial H}{\partial T}\right)_P dT + \frac{1}{T}\left[\left(\frac{\partial H}{\partial P}\right)_T - V\right]dP$$

$$\left[\frac{\partial}{\partial P}\frac{1}{T}\left(\frac{\partial H}{\partial T}\right)_P\right]_T = \left[\frac{\partial}{\partial T}\frac{1}{T}\left[\left(\frac{\partial H}{\partial P}\right)_T - V\right]\right]_P$$

$$0 = -\frac{1}{T}\left(\frac{\partial V}{\partial T}\right)_P - \frac{1}{T^2}\left[\left(\frac{\partial H}{\partial P}\right)_T - V\right]$$

$$\Rightarrow \quad 0 = \left(\frac{\partial H}{\partial P}\right)_T = V - T\left(\frac{\partial V}{\partial T}\right)_P \quad \Rightarrow \quad \left(\frac{\partial V}{\partial T}\right)_P = \frac{V}{T}$$

$$0 = \left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P \quad \Rightarrow \quad \left(\frac{\partial P}{\partial T}\right)_V = \frac{P}{T}$$

$$dP = \left(\frac{\partial P}{\partial V}\right)_T dV + \left(\frac{\partial P}{\partial T}\right)_V dT \qquad P = \text{const.} \Rightarrow 0 = \left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P + \left(\frac{\partial P}{\partial T}\right)_V$$

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{\left(\frac{\partial P}{\partial T}\right)_V}{\left(\frac{\partial V}{\partial T}\right)_P} = -\frac{\frac{P}{T}}{\frac{V}{T}} = -\frac{P}{V}$$

$$dP = \left(\frac{\partial P}{\partial V}\right)_T dV + \left(\frac{\partial P}{\partial T}\right)_V dT$$

$$dP = -\frac{P}{V}dV + \frac{P}{T}dT$$

$$\frac{dP}{P} = -\frac{dV}{V} + \frac{dT}{T} \quad \Rightarrow \quad \ln P = \ln\frac{T}{V} + \ln C \quad \Rightarrow$$

$$\Rightarrow \quad \underline{\frac{PV}{T} = C}$$

4. (2.) $M = a\frac{\mathcal{H}}{T}$ Curie $\Rightarrow M_1 \to M_2$ $\Delta S = ?$

T, V const. tokom $M_1 \to M_2$

$$TdS = dU - \mathcal{H}dM$$

jen. energije

$$\left(\frac{\partial U}{\partial M}\right)_T = -T\left(\frac{\partial \mathcal{H}}{\partial T}\right)_M + \mathcal{H} = -T\cdot\frac{M}{a} + \frac{MT}{a} = 0$$

$V = \text{const.}$

$U = U(T, M)$ & $T = \text{const.}$

$\Rightarrow dU = 0$

$$dS = -\frac{\mathcal{H}}{T}dM$$

$$dS = -\frac{M}{a}dM$$

$$S\Big|_{S_1}^{S_2} = -\frac{1}{2a}M^2\Big|_{M_1}^{M_2} \Rightarrow S_2 - S_1 = \frac{1}{2a}\left(M_1^2 - M_2^2\right)$$

5. (1.) $U = AVT^n$ $n \neq 1$ $\Rightarrow G = 0$

$$dS = \frac{1}{T}dU + \frac{P}{T}dV \qquad G = U + PV - TS$$

$$dU = nAVT^{n-1}dT + AT^n dV$$

$$\Rightarrow dS = nAVT^{n-2}dT + \left(AT^{n-1} + \frac{P}{T}\right)dV$$

$$\left[\frac{\partial}{\partial V}\left(nAVT^{n-2}\right)\right]_T = \left[\frac{\partial}{\partial T}\left(AT^{n-1} + \frac{P}{T}\right)\right]_V$$

$$nAT^{n-2} = (n-1)AT^{n-2} - \frac{P}{T^2} + \frac{1}{T}\left(\frac{\partial P}{\partial T}\right)_V \qquad P = P(T, V)$$

$$AT^{n-2} = -\frac{P}{T^2} + \frac{1}{T}\left(\frac{\partial P}{\partial T}\right)_V \quad /\cdot T^2 \qquad AT^{n-2} = \frac{\partial}{\partial T}\left(\frac{P}{T}\right)_V$$

$$\underline{AT^n = -P + T\left(\frac{\partial P}{\partial T}\right)_V} \qquad \frac{P}{T} = \frac{A}{n-1}T^{n-1} + f(V)$$

$$P = \frac{A}{n-1}T^n + Tf(V)$$

Opšte rešenje: Euler dif. jen. $\sum_{n=1}^{N} x^n \frac{d}{dx^n}(y(x)) = f(x)$

$T > 0 \Rightarrow T = e^t \Rightarrow \left(\frac{\partial}{\partial T}\right)_V = \left(\frac{\partial t}{\partial T}\right)_V\left(\frac{\partial}{\partial t}\right)_V = \frac{1}{T}\left(\frac{\partial}{\partial t}\right)_V \Rightarrow Ae^{nt} = \left(\frac{\partial P}{\partial t}\right)_V - P$

$t = \ln T$

$\Rightarrow P = C\cdot e^{rt}$ $\quad r - 1 = 0$ $\quad r = 1$ $\quad \Rightarrow P_h = Ce^t = C\cdot T = \underline{f(V)\cdot T}$

za $n \neq 1$ $\quad P_p = Ce^{nt} \Rightarrow Ae^{nt} = Cne^{nt} - Ce^{nt} \Rightarrow A = C(n-1) \Rightarrow C = \frac{A}{n-1}$

$$P = \frac{A}{n-1}T^n + Tf(V)$$

$$P = \frac{A}{n-1} T^{n} + \underbrace{T f(V)}_{\text{nonaq. rjes}}$$

$$dS = nAVT^{n-2}dT + \left[AT^{n-1} + \frac{A}{n-1}T^{n-1} + f(V)\right]dV$$

$$dS = nAVT^{n-2}dT + \left[\frac{n}{n-1}AT^{n-1} + f(V)\right]dV$$

tot. dif. $\Rightarrow$ eqz dif. jba. $\quad z = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy =$

$$S = \int_{T_0}^{T} nAVT^{n-2}dT + \int_{V_0}^{V}\left[\left(\frac{n}{n-1}\right)AT^{n-1} + f(V)\right]dV$$

$$S = \left(\frac{nAV}{n-1}T^{n-1}\right)_{T_0}^{T} + \left(\frac{n}{n-1}AT_0^{n-1}\cdot V\right)_{V_0}^{V} + \int_{V_0}^{V} f(V)\,dV$$

$$S = \frac{nAV}{n-1}T^{n-1} + \frac{nAV}{n-1}T_0^{n-1} + \frac{n}{n-1}AT_0^{n-1}\cdot V + \frac{n}{n-1}AT_0^{n-1}V_0 + \int_{V_0}^{V} f(V)\,dV$$

$$S = \frac{nT^{n-1}}{n-1}AV + \underline{\underline{\frac{2nA}{n-1}T_0^{n-1}\cdot V}} + \frac{nAV_0}{n-1}T_0^{n-1} + \underline{\underline{\int_{V_0}^{V} f(V)\,dV}}$$

$$G = U + PV - TS = AVT^{n} + \frac{A}{n-1}T^{n}V + TVf(V) - \frac{n}{n-1}T^{n}AV - \frac{2nAT_0^{n-1}}{n-1}TV$$

$$\underline{G} = AVT^{n}\left(1 + \frac{1}{n-1} - \frac{n}{n-1}\right) = 0$$

6. Odrediti ovisnost volumena od energije za van der Waals-ov plin

$$U(V) = ?$$

pretp: $U(T,V)$ $\qquad \left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT$

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P \qquad P = \frac{nRT}{V-nb} - \frac{an^2}{V^2}$$

$$\Rightarrow \left(\frac{\partial U}{\partial V}\right)_T = T\cdot\frac{nR}{V-nb} - \frac{nRT}{V-nb} + \frac{an^2}{V^2} = \frac{an^2}{V^2}$$

$$\Rightarrow \underline{U(T,V) = U(T) - \frac{an^2}{V}}$$

Iz. 1. Izračunati

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \qquad \chi = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

$$\beta = \frac{1}{P}\left(\frac{\partial P}{\partial T}\right)_V$$

$$P\cdot\beta = -\left(\frac{\partial P}{\partial T}\right)_V = -\left(\frac{\partial P}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_P = -\frac{\left(\frac{\partial V}{\partial T}\right)_P}{\left(\frac{\partial V}{\partial P}\right)_T} = -\frac{\alpha}{\chi}$$

$$C_p - C_v = PVT\alpha\beta = \frac{TV\alpha^2}{\chi}$$

7. Odrediti $C_p - C_v$ za exp. dobiv.

$$P = \frac{RT}{V}\left[1 + \frac{A}{VT^{3/2}}\right] \qquad \frac{A}{VT^{3/2}} \ll 1$$

$$C_p - C_v = T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_P$$

$$\underline{\left(\frac{\partial P}{\partial T}\right)_V} = \frac{R}{V}\left[1 + \frac{A}{VT^{3/2}}\right] + \frac{RT}{V}\left[0 - \frac{3}{2}\frac{A}{VT^{5/2}}\right] =$$

$$= \frac{R}{V}\left[1 + \frac{A}{VT^{3/2}} + \cancel{1} - \frac{3}{2}\frac{A}{VT^{3/2}}\right] = \underline{\frac{R}{V}\left[1 + \cancel{1} - \frac{1}{2}\frac{A}{VT^{3/2}}\right]}$$

još $\left(\frac{\partial V}{\partial T}\right)_P = ?$

$$\left(\frac{\partial U}{\partial V}\right)_T + P = T\left(\frac{\partial P}{\partial T}\right)_V$$

$$\left(\frac{\partial U}{\partial V}\right)_T = \frac{RT}{V}\left[1 - \frac{1}{2}\frac{A}{VT^{3/2}}\right] - \frac{RT}{V}\left[1 + \frac{A}{VT^{3/2}}\right] = \frac{RT}{V}\left(-\frac{3}{2}\right)\cdot\frac{A}{VT^{3/2}}$$

$$\Rightarrow U = +\frac{3}{2}\frac{RT}{V}\cdot\frac{A}{T^{3/2}} + \underbrace{U(T)}_{U_{idealni}} = \frac{3}{2}RT \quad \text{za } 1-A$$

$$\underline{U = \frac{3}{2}RT\left(1 + \frac{A}{VT^{3/2}}\right)}$$

$$PV^2 = RTV + RT\frac{A}{T^{3/2}}$$

$$V^2 = \frac{RT}{P}V + \frac{RT}{P}\cdot\frac{A}{T^{3/2}} \Rightarrow V^2 - \frac{RT}{P}V - \frac{RA}{PT^{1/2}} = 0$$

$$V_{12} = \frac{RT}{2P} \pm \sqrt{\frac{R^2T^2}{4P^2} + \frac{RA}{PT^{1/2}}}$$

nedolazi u obzir jer $V<0$

$$\frac{A}{VT^{3/2}} \ll 1$$

$$P^2 = \frac{R^2T^2}{V^2}\left[1 + \frac{2A}{VT^{3/2}}\right]$$

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{R}{2P} + \frac{1}{2}\,\frac{\frac{R^2}{4P^2}\cdot 2T - \frac{1}{2}\frac{RA}{PT^{3/2}}}{\sqrt{\frac{R^2T^2}{4P^2} + \frac{RA}{PT^{1/2}}}} = \frac{R}{2P}$$

$$= \frac{R}{2P} + \frac{\frac{R^2T}{4P^2} - \frac{RA}{4PT^{3/2}}}{\sqrt{\frac{R^2T^2}{4P^2} + \frac{RA}{PT^{1/2}}}}$$

ili $V^2 - \frac{RT}{P}V - \frac{RA}{PT^{1/2}} = 0 \quad / \left(\frac{\partial}{\partial T}\right)_P$

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{RV - \frac{1}{2}\frac{AR}{T^{3/2}}}{2PV - RT} = \frac{RV - \frac{1}{2}\frac{AR}{T^{3/2}}}{2RT + \frac{2ART}{VT^{3/2}} - RT}$$

$$= \frac{V}{T}\cdot\frac{1 - \frac{1}{2}\frac{A}{VT^{3/2}}}{1 + 2\frac{A}{VT^{3/2}}} \approx \frac{V}{T}\left(1 - \frac{1}{2}\frac{A}{VT^{3/2}}\right)\left(1 - 2\frac{A}{VT^{3/2}}\right)$$

$$\approx \frac{V}{T}\left(1 - \frac{5}{2}\frac{A}{VT^{3/2}}\right) \quad \text{ano} \quad \frac{R}{V}\left[1 - \frac{1}{2}\frac{AP}{VT^{3/2}}\right]$$

$$\Rightarrow C_P - C_V = \cancel{T}\cdot\frac{R}{\cancel{V}}\left(1 - \frac{1}{2}\frac{A}{VT^{3/2}}\right)\cdot\frac{\cancel{V}}{\cancel{T}}\left(1 - \frac{5}{2}\frac{A}{VT^{3/2}}\right)$$

$$\approx R\left(1 - 3\frac{A}{VT^{3/2}}\right)$$

Statistička fizika

Integrali:

$$I_n = \int_0^\infty x^n e^{-x^2} dx = \int_0^\infty x^{n-1} e^{-x^2} \cdot \frac{1}{2} d(x^2) = -\frac{1}{2} \int_0^\infty x^{n-1} d(e^{-x^2}) =$$

$$= \underbrace{\left[-\frac{1}{2} x^{n-1} e^{x^2}\right]_0^\infty}_{0} + \frac{n-1}{2} \int_0^\infty x^{n-2} e^{-x^2} dx$$

$$\Rightarrow \quad I_n = \frac{n-1}{2} I_{n-2} = \frac{n-1}{2} \cdot \frac{n-3}{2} \cdot I_{n-4}$$

$n = 2K$

$$I_{2K} = \underbrace{\frac{2K-1}{2} \cdot \frac{\overset{(2\cdot 2-1)}{2K-3}}{2} \cdots \frac{\overset{2K-(2K-1)}{1}}{2}}_{K\times} I_0 \qquad I_0 = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$n = 2K+1$

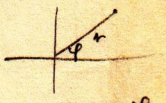
$$I_{2K+1} = \frac{(2K+1)-1}{2} \cdot \frac{(2K+1)-3}{2} \cdots \frac{\overset{(2K+1)-(2K-1)}{2}}{2} I_{(2K+1)-2K}$$

$$= K(K-1) \cdots 1 \; I_1 \qquad I_1 = \int_0^\infty x e^{-x^2} dx = -\frac{1}{2} \int_0^\infty d(e^{-x^2}) = \frac{1}{2}$$

$$\Rightarrow \quad I_{2K+1} = \frac{1}{2} K!$$

→ Blanuša strogo!

$$I_0 = \int_0^\infty e^{-x^2} dx = \frac{1}{2} \int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\frac{1}{4} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} dx\, dy}$$



$x = r\cos\varphi$
$y = r\sin\varphi$

$$\frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix} = r$$

$$I_0^2 = \frac{1}{4} \int_0^\infty \int_0^{2\pi} \rho e^{-\rho^2} d\rho\, d\varphi = \frac{\pi}{2} \int_0^\infty \rho e^{-\rho^2} d\rho$$

$$= \frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4}$$

$$\Rightarrow \quad I_0 = \frac{\sqrt{\pi}}{2}$$

Preko Γ-fje.

$$\Gamma(x) \overset{def.}{=} \int_0^\infty e^{-y} y^{x-1} dy \qquad \underline{\Gamma(x+1) = x\Gamma(x)} \qquad \text{dokaz. za } x>0$$

$$e^{-t} t^{x-1} dt$$

$$\underline{I_n} = \int_0^\infty x^n e^{-x^2} dx = \frac{1}{2}\int_0^\infty t^{\overset{\left(\frac{n}{2}+\frac{1}{2}-1\right)}{\frac{n-1}{2}}} e^{-t} dt = \underline{\frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)} = \frac{1}{2}\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)$$

$$t = x^2 \Rightarrow dx = \frac{1}{2} t^{-\frac{1}{2}} dt$$

$$x = t^{\frac{1}{2}}$$

$$\underline{I_n} = \frac{1}{2}\cdot\left(\frac{n}{2}-\frac{1}{2}\right)\cdot\Gamma\left(\frac{n}{2}-\overset{-1+1}{\frac{1}{2}}\right) =$$

$$= \underline{\frac{1}{2}\,\frac{n-1}{2}\cdot\frac{n-3}{2}\,\Gamma\left(\frac{n-3}{2}\right)} \qquad \text{itd.}$$

$$\Rightarrow \underline{I_n = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)} \qquad \text{gdje} \quad \boxed{\begin{array}{l}\Gamma(n+1) = n! \\ \Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi}\end{array}}$$

$$\Gamma(1) = \int_0^\infty e^{-y} dy = -e^{-y}\Big|_0^\infty = 1$$

$$\text{jer} \quad \Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = 2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$$

primjeri :

$$\int_0^\infty x^3 e^{-x^2} dx = \frac{1}{2}\Gamma\left(\frac{4}{2}\right) = \frac{1}{2}\Gamma(2) = \frac{1}{2}$$

$$\int_0^\infty x^4 e^{-x^2} dx = \frac{1}{2}\Gamma\left(\frac{5}{2}\right) = \frac{1}{2}\cdot\frac{4!}{2^4\cdot 2!}\sqrt{\pi} = \frac{\cancel{4}\cdot 3\cdot \cancel{2}\cdot 1}{2^{\cancel{4}}\cdot \cancel{2}}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$$

Elementi vjerojatnosti

def. prema exp.

m uzastopnih mjerenja veličine F daje N različ. vrijednosti.

P_1 puta F_1

$\vdots$

P_i puta F_i

$\vdots$

P_N puta F_N

$$\sum_{i=1}^{N} P_i = m$$

vjerojatnost veličine F_i

$$W_i = \frac{P_i}{m} \xrightarrow[m\to\infty]{} \text{const.} \qquad W_i = \frac{P_i}{\sum_{i=1}^{N} P_i}$$

$$\sum_{i=1}^{N} W_i = \frac{1}{m}\sum_{i=1}^{N} P_i = 1$$

F_i ili F_j (isključivi, nespojivi događaji)

$$W(i+j) = \frac{P_i + P_j}{m} = W_i + W_j$$

ili - ili

Dva sistema

- mjerimo na da dva sistema

A B

F_A u A, F_B u B

$$W_A = \frac{P_A}{m_A} \qquad W_B = \frac{P_B}{m_B}$$

F_A i F_B (nezavisni događaji)

da se u A pojavi F_A, a u B F_B

$$W(A,B) = \frac{P_A \cdot P_B}{m_A \cdot m_B} = W_A \cdot W_B$$

i - i

4 - 13 Karata $\quad W(A\ pik) = \frac{1}{4}\cdot\frac{1}{13}$

20 Karata
4 asa
1 pik. A

$W(A\ pik) = \frac{4}{20}\cdot\frac{1}{4}$ ili $\frac{5}{20}\cdot\frac{1}{5}$

Fja - raspodjele

Prebrojavanje ~~rezul~~ vjerojatnosti za različite ishode rezultata m uzast. mjerenja F daje N različ. vrij.

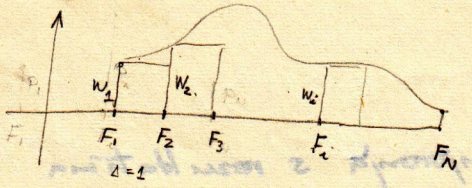
$P_1 \dots F_1$

$\vdots$

$P_i \dots F_i$

$\vdots$

$P_N \dots F_N$



$$m_i = \sum_{i=1}^{N} P_i$$

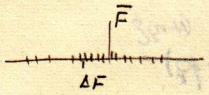
$$\bar{F} = \langle F \rangle = \sum_{i=1}^{N} F_i W_i$$

disperzija

$$\overline{(\Delta F)^2} = \overline{(F-\bar{F})^2} = \overline{F^2} - \bar{F}^2$$

$-2\bar{F}^2 + \bar{F}^2$

$$= \sum_{i=1}^{N} (F_i - \bar{F})^2 W_i = \sum_{i=1}^{N} F_i^2 W_i - 2\sum_{i=1}^{N} \bar{F} F_i W_i + \bar{F}^2 \sum_{i=1}^{N} W_i$$



$$\overline{(F+G)} = \bar{F} + \bar{G}$$

→ dokaz Hinčin

$$\overline{F \cdot G} \neq \bar{F} \cdot \bar{G}$$ općenito

$$\overline{F \cdot G} = \bar{F} \cdot \bar{G}$$

ako se te velič. mjere nezav. mjerilima

jedino ako se mjere nezavisno, kad $W = W_F \cdot W_G$

Binomna raspodjela

pismo

O ⊗ O ⊗ ⊗ O O ⊗ ... O

glava

N - pokusaja

p - vjeroj. u 1 bacanju pojave glave

$q = 1 - p$ - vjeroj. u 1 bac. pojave pisma

pri je to potpun skup događaja

n x glava u N pokušaja:

broj. načina $\dfrac{N!}{n!\,(N-n)!}$

$$P_n(N) = A \frac{N!}{n!\,(N-n)!} p^n \cdot q^{N-n}$$

vjerojatnost da od N-pokušaja n-puta padne glava

ili - ili vjerojatnost

normiranost ? $\sum_{n=0}^{N} P_n(N) \overset{?}{=} 1$ ... u N pokušaja

$$\underline{\sum_{n=0}^{N} A P_n(N)} = \sum_{n=0}^{N} \frac{N!}{n!\,(N-n)!} p^n q^{N-n} = \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} \underline{= A(p+q)^N = 1}$$

$\Rightarrow A = 1$

Iz :

① N različ. mjerenja s rezultatima $P_1, P_2, \dots P_k$

Kolika je vjerojatnost

$$P_{n_1, n_2, \dots n_k}(N) = ?$$

P_1 pojavi n_1 puta

$\vdots$

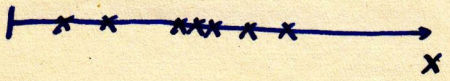
P_k pojavi n_k puta

(polinomna raspodjela)

○ ⊗ ○ ⊗ ○○⊗ . . . ○

$$P_{n_1, \dots n_k}(N) = \frac{N!}{n_1! \dots n_k! \,(N - \sum_{k=1}^{k} n_k)!} \prod_{k=1}^{k} p_k^{n_k} (1-p_k)^{(N-n_k)}$$

② Mjerimo vrem. interval otkucaja u Geig-Mil. brojaču → vrem. točke ili točka u kojima se molek. plina sudara po nekoj trajektoriji



$[\frac{1}{\lambda}] = 1$ točka $/m$

pretp. : srednja gustoća tih točaka $\frac{1}{\lambda}$ ($\frac{1}{\lambda}$ točaka po 1m)

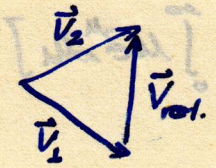
$\lambda(m) \to 1$ točka

→ vjerojatnost da točka bude u intervalu širine dx

$$W_{poj}(dx) = W_{dx} = \frac{dx}{\lambda}$$

Kolika je vjerojatnost $P_0(\frac{x}{\lambda}) = ?$ da na intervalu dužine x ne nađe ni jedna točka

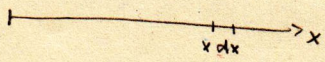
③ Mnoštvo čestica se giba brzinom istog iznosa, ali različitog smjera. Kolika je prosječna relativna brzina čestica



$$\overline{V_{rel}} = \overline{|\vec{V}_1 - \vec{V}_2|}$$

Rješ. Dz.

② Exponencijalna raspodjela



$\frac{1}{\lambda}$ sr. broj čestica po 1m

$P_0\left(\frac{x}{\lambda}\right)$ vjeroj. da na x ne nađemo niti jednu česticu

$P_0\left(\frac{x+dx}{\lambda}\right)$ vjer. da na x+dx ne nađemo niti jednu česticu

Taylor-ov red
$$P_0\left(\frac{x+dx}{\lambda}\right) = P_0\left(\frac{x}{\lambda}\right) + \frac{dx}{1!}\cdot\frac{d}{dx}P_0\left(\frac{x}{\lambda}\right)_{dx=0} + \frac{(dx)^2}{\dots} + \dots = P_0\left(\frac{x}{\lambda}\right) + dx\cdot\frac{d}{dx}P_0\left(\frac{x}{\lambda}\right)$$

rezav. događ.
$$P_0\left(\frac{x+dx}{\lambda}\right) = \underbrace{P_0\left(\frac{x}{\lambda}\right)}_{\text{ne nađemo na } x}\cdot\underbrace{\left(1-\frac{dx}{\lambda}\right)}_{\text{ne nađemo na } x,x+dx}$$

W_{dx} da nađemo

$$\Rightarrow \quad -\frac{\cancel{dx}}{\lambda}P_0\left(\frac{x}{\lambda}\right) = \cancel{dx}\frac{d}{dx}P_0\left(\frac{x}{\lambda}\right)$$

$$\frac{d\,P_0\left(\frac{x}{\lambda}\right)}{P_0\left(\frac{x}{\lambda}\right)} = -\frac{dx}{\lambda} \qquad \ln P_0\left(\frac{x}{\lambda}\right) = -\frac{x}{\lambda} + \ln C \Rightarrow P_0\left(\frac{x}{\lambda}\right) = Ce^{-\frac{x}{\lambda}}$$

$$1 = P_0\left(\frac{0}{\lambda}\right) = C$$

$$\Rightarrow \quad P_0\left(\frac{x}{\lambda}\right) = e^{-\frac{x}{\lambda}}$$

→ Gustoća vjerojatnosti: dx = vjerojatnost da od dane čestice koju prvu nađemo je između x i x+dx

samo na x, x+dx idući od 0 → x

$$dW_{poj}(x, x+dx) = \rho(x)dx = \underbrace{P_0\left(\frac{x}{\lambda}\right)}_{\text{na x ne}}\cdot\underbrace{\frac{dx}{\lambda}}_{\text{da nađemo na dx}} = \frac{1}{\lambda}P_0\left(\frac{x}{\lambda}\right)dx$$

$\rho(x) = \frac{1}{\lambda}P_0\left(\frac{x}{\lambda}\right)$ gustoća vjeroj. da prvu česticu koju nađemo je u x, x+dx, ne općenito pojave na x, x+dx bez obzira na druge čestice

$\bar{x}$ - srednja dužina da nađemo točku

$\frac{x}{\lambda}=u \quad dx=\lambda du$

$$\bar{x} = \int_0^\infty x\,\rho(x)dx = \int_0^\infty x\cdot\frac{1}{\lambda}e^{-\frac{x}{\lambda}}dx = \frac{1}{\lambda^2}\left[\int_0^\infty u e^{-u}du\right] = \lambda\left[-ue^{-u}\right]_0^\infty + \int_0^\infty e^{-u}du$$

$$= \lambda\cdot\left[-e^{-u}\right]_0^\infty = \lambda$$

$$\underline{\bar{x} = \lambda}$$

Kvadratno odstupanje :

$\frac{x}{\lambda}=u \quad dx=\lambda du \quad \int_0^\infty \lambda^2u^2\cdot\frac{1}{\lambda}e^{-u}\cdot\lambda du = \lambda^2\int_0^\infty u^2e^{-u}du$

$= \lambda^2 e^{-u}\left[-u^2-2u-2\right]$

$= 2\lambda^2$

$$(\Delta x)^2 = \overline{(x-\bar{x})^2} = \overline{x^2} - \bar{x}^2 = \int_0^\infty x^2\cdot\frac{1}{\lambda}e^{-\frac{x}{\lambda}}dx - \lambda^2$$

$$(\Delta x)^2 = \lambda^2$$

$x-\bar{x}$ - odstupanje u smislu da se prva točka pojavi na x a ne na $\bar{x}$

Poisson-ova raspodjela

→ Kolika je vjerojatnost da se u interv. dužine x nađe n točaka

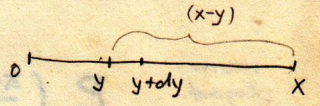
vjerojatnost nalaženja :

sumacija po x ili-ili događaja (y_1, y_1+dy_1 ili y_2, y_2+dy_2 ...)

$n=1$ (1-točke)

$$\underline{P_1\left(\frac{x}{\lambda}\right)} = \int_0^x \rho(y)dy\cdot\underbrace{P_0\left(\frac{x-y}{\lambda}\right)}_{\text{u ostatku nema točke}}$$

bilo gdje u x



$$P_0\left(\frac{x-y}{\lambda}\right) = e^{-\frac{x-y}{\lambda}} = e$$

$$\rho(y)dy = \frac{1}{\lambda}P_0\left(\frac{y}{\lambda}\right)dy$$

vjerojatnost da točke nema u $(0,y)$ već u $(y,y+dy)$

$$P_0\left(\frac{x-y}{\lambda}\right) = e^{-\frac{x-y}{\lambda}}$$

vjerojatnost da točke nema na intervalu (y^{+dy}, x)

$x-y=0 \Rightarrow x=y$ kad $x\in(y,x)$ $\quad x-y\in(0,x-y)$

$$\Rightarrow \rho(y)dy\,P_0\left(\frac{x-y}{\lambda}\right) = \frac{1}{\lambda}e^{-\frac{y}{\lambda}}dy\cdot e^{-\frac{x-y}{\lambda}}$$

vjerojatnost da točke nema u $(0,y)$ & $(y+dy,x)$ već samo u $(y,y+dy)$

$$= \int_0^x \frac{1}{\lambda}\cdot e^{-\frac{x}{\lambda}}dy = \underline{\frac{x}{\lambda}e^{-\frac{x}{\lambda}}}$$

$$\int_0^x P_1\left(\frac{y}{\lambda}\right)\frac{dy}{\lambda}\cdot P_0\left(\frac{x-y}{\lambda}\right) = \int_0^x \frac{y}{\lambda}e^{-\frac{y}{\lambda}}\cdot\frac{dy}{\lambda}\cdot e^{-\frac{x-y}{\lambda}} = \frac{e^{-\frac{x}{\lambda}}}{\lambda^2}\int_0^x y\,dy$$

? ispitaj drugu mogućnost

$n=2$ (2-točke)

$$\underline{P_2\left(\frac{x}{\lambda}\right)} = \int_0^x \rho(y)dy\cdot\underbrace{P_1\left(\frac{x-y}{\lambda}\right)}_{\text{vjerojatnost pojave 1 točke u } (y+dy,\,x)}$$

$$= \int_0^x \frac{1}{\lambda}e^{-\frac{y}{\lambda}}dy\cdot\frac{x-y}{\lambda}e^{-\frac{x-y}{\lambda}} = \frac{1}{\lambda^3}\int_0^x e^{-\frac{x}{\lambda}}(x-y)dy$$

$xy-\frac{y^2}{2}$

$$= \underline{\frac{1}{2}\frac{x^2}{\lambda^2}e^{-\frac{x}{\lambda}}}$$

$x-y=u \quad du=-dy$

$$\frac{e^{-\frac{x}{\lambda}}}{\lambda^3}\int_0^x\frac{(x-y)^2}{2}dy = \frac{e^{-\frac{x}{\lambda}}}{2!\,\lambda^3}\left(-\int_x^0 u^2dy\right) = \frac{e^{-\frac{x}{\lambda}}}{3!\,\lambda^3}x^3$$

tot. nd. $\Rightarrow$ $$\underline{P_n\left(\frac{x}{\lambda}\right) = \frac{1}{n!}\left(\frac{x}{\lambda}\right)^n e^{-\frac{x}{\lambda}}}$$ Poisson-ova raspodjela

→ vjerojatnost da na dužini x nađemo bilo koji broj čestica uključujući 0

$$\sum_{n=0}^{\infty} P_n\left(\frac{x}{\lambda}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x}{\lambda}\right)^n e^{-\frac{x}{\lambda}} = e^{-\frac{x}{\lambda}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x}{\lambda}\right)^n = e^{-\frac{x}{\lambda}} \cdot e^{\frac{x}{\lambda}} = 1$$

→ srednji broj čestica na dužini x

$$\bar{n} = \sum_{n=0}^{\infty} n P_n\left(\frac{x}{\lambda}\right) = \sum_{n=1}^{\infty} \frac{n}{n!}\left(\frac{x}{\lambda}\right)^n e^{-\frac{x}{\lambda}} = \frac{x}{\lambda} \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\left(\frac{x}{\lambda}\right)^{n-1} e^{-\frac{x}{\lambda}} = \frac{x}{\lambda}$$

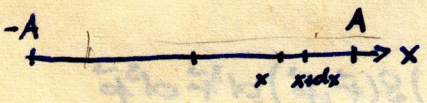
jer član otpada
$0 \cdot P_0\left(\frac{x}{\lambda}\right)$

Problem:

① Odrediti gustoću vjerojatnosti za harmonijski oscilator

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 \quad \Rightarrow \quad \ddot{x} + \omega^2 x = 0$$

$x = A\sin(\omega t + \alpha)$
$\dot{x} = A\omega\cos(\omega t + \alpha)$



vjeroj. da se čest. nađe u $(x, x+dx)$

$\rho(x)dx$

1° ta vjeroj. prop. vrem. u kojem se čestica zadržava u $(x, x+dx)$

$$\rho(x)dx = k\,dt$$

$$E = H = \frac{m}{2} A^2\omega^2\cos^2(\omega t+\alpha) + \frac{m\omega^2}{2} A^2\sin^2(\omega t+\alpha) = \frac{m\omega^2}{2} A^2$$

$$\Rightarrow A = \sqrt{\frac{2E}{m\omega^2}}$$

$$\int_{-A}^{A} \rho(x)dx = 1 \quad \text{za } \alpha = -\frac{\pi}{2} \quad \rightarrow \quad \int_0^{T/2} k\,dt = 1 \quad \Rightarrow k = \frac{2}{T} = \frac{\omega}{\pi}$$

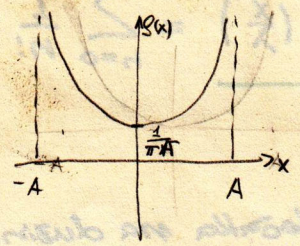
$dx \quad x = A\sin(\omega t + \frac{\pi}{2})$
$\frac{2\pi}{T} \quad \frac{T}{2}$

$x = A\sin(\omega t + \alpha)$

$\cos(\omega t + \alpha) = \sqrt{1 - \frac{x^2}{A^2}}$

$$\rho(x) = k\frac{dt}{dx} = \frac{\omega}{\pi}\cdot\frac{1}{\dot{x}} = \frac{\omega}{\pi}\cdot\frac{1}{A\omega\cos(\omega t+\alpha)}$$

$$\Rightarrow \rho(x) = \frac{1}{\pi\sqrt{A^2 - x^2}}$$



$$\rho(x) = \frac{1}{\pi}\cdot\frac{1}{\sqrt{\frac{2E}{m\omega^2} - x^2}}$$

u kvantnoj meh. npr za $\pm A$

$\rho(x) = \infty$

II z.

④ Izračunati $\Delta x \cdot \Delta p$ za harm. oscilator

$$\Delta x = \left(\overline{x^2} - \bar{x}^2\right)^{1/2}$$

opć. $\overline{f(x)} = \int_{-A}^{A} f(x)\,\rho(x)\,dx$

čestice s klas. svojstvima :

$$\rho(\vec{x}, \vec{p}) \qquad \bar{f} = \int f(\vec{r}, \vec{p})\,\rho(\vec{r}, \vec{p})\,d^3\vec{r}\,d^3\vec{p}$$

ako f ovisi o iznosu $\Rightarrow$ transf. koordinata

⑤ Pokazati da je fazni prostor invarijantan na prijelaz iz Kartez. u sferni sustav

$p_i = \frac{\partial T}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}$

$$dx\,dy\,dz\,dp_x\,dp_y\,dp_z = r^2\sin\theta\,dr\,d\theta\,d\varphi\,dp_r\,dp_\theta\,dp_\varphi$$

pokazano na 26

⑥ Fazni volumen je invarijantan na kanonske transformacije

$$d\phi = d^3\vec{r}\,d^3\vec{p}$$

$$x = x(Q_1, Q_2, Q_3)$$
$$P_i = P_i(P_1, P_2, P_3)$$

pokazano na 26 b)

– time je dokazan i Liouvil. tm. da su fazni volum. jednako apriori vjerojatni

Рješ: II2. & ③ str 7.

④

$$\rho(x) = \frac{1}{\pi}\cdot\frac{1}{\sqrt{A^2-x^2}} \qquad A=\sqrt{\frac{2E}{m\omega^2}} \qquad \Delta x\cdot\Delta p = ?$$

$$\overline{\Delta x^2} = \overline{(x-\bar{x})^2} = \overline{x^2}-\bar{x}^2 = \overline{(x^2-\bar{x}^2)}$$

$$\sqrt{\overline{\Delta x^2}} = \Delta x = (\overline{x^2}-\bar{x}^2)^{1/2}$$

$u = A^2-x^2 \quad du = -2x\,dx$

$$\bar{x} = \int_{-A}^{A} x\,\rho(x)\,dx = \frac{1}{\pi}\int_{-A}^{A}\frac{x}{\sqrt{A^2-x^2}}\,dx = -\frac{1}{2\pi}\int_0^0 u^{-1/2}\,du = -\frac{1}{2\pi}\frac{1}{-\frac{1}{2}+1}\,u^{1/2}$$

$$= -\frac{1}{\pi}\sqrt{A^2-x^2}\,\Big|_{-A}^{A} = 0$$

$x^2 = A^2\sin^2(\omega t+\alpha) \quad (\alpha = -\frac{\pi}{2})$ $\qquad \rho(x)\,dx = \frac{\omega}{\pi}dt$

$$= \frac{\omega A^2}{\pi}\left[\frac{\omega t+\alpha}{2\omega} - \frac{1}{4\omega}\sin 2(\omega t+\alpha)\right]_0^{T/2}$$

$$\overline{x^2} = \int_0^{T/2} A^2\sin^2(\omega t+\alpha)\,\rho(x)\,dx = \frac{\omega A^2}{\pi}\int_0^{T/2}\sin^2(\omega t+\alpha)\,dt = \frac{\omega A^2}{\pi}\left(\cancel{\frac{1}{2}t - \frac{1}{4\omega}\sin 2(\omega t+\alpha)}\right)$$

$\frac{2\pi}{T}$ $\qquad \frac{2\pi}{T}t$ $\quad 2\omega t+2\alpha$

$$= \frac{\omega A^2}{\pi}\left[\frac{T}{4} - \frac{1}{4\omega}\left(\sin\left(2\cdot\frac{2\pi}{T}\cdot\frac{T}{2}+2\alpha\right) - \sin 2\alpha\right)\right] = \frac{2\pi A^2}{\pi T}\frac{T}{4} = \frac{A^2}{2}$$

$$\overline{x^2} = \frac{A^2}{2} = \frac{E}{m\omega^2}$$

$$\Rightarrow \Delta x = \sqrt{\frac{E}{m\omega^2}} = \frac{1}{\omega}\sqrt{\frac{E}{m}}$$

$\frac{2\pi}{T}\cdot\frac{T}{2} = \pi$

$$\Delta p = (\overline{p^2}-\bar{p}^2)^{1/2} \qquad \bar{p} = \int_0^{T/2} mA\omega\cos(\omega t+\alpha)\cdot\frac{\omega}{\pi}dt = 0$$

$m\dot{x}$ $\qquad \sin(\pi+\alpha) - \sin\alpha = 0$

$$E = H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

$$E = \bar{E} = \frac{1}{2m}\overline{p^2} + \frac{\cancel{m\omega^2}}{2}\cdot\frac{E}{\cancel{m\omega^2}} \Rightarrow \overline{p^2} = mE$$

$$\Rightarrow \Delta p = \sqrt{mE}$$

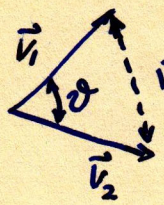
$$\Rightarrow \Delta x\cdot\Delta p = \frac{E}{\omega}$$

U kvantnoj mehanici:

$$E = n\hbar\omega \Rightarrow \Delta x\cdot\Delta p = \hbar$$

Dz.

③ sa 7.



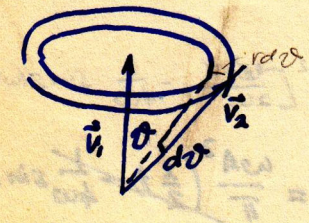
$\vec{v}_r = \vec{v}_1 - \vec{v}_2$

$$|\vec{v}_1| = |\vec{v}_2| = v$$

$$\bar{v}_{rel} = \overline{|\vec{v}_1 - \vec{v}_2|}$$

$$v_r = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\vartheta} = v\sqrt{2(1-\cos\vartheta)} = 2v\sin\frac{\vartheta}{2}$$

$$\Rightarrow \quad \bar{v}_r = 2v\,\overline{\sin\frac{\vartheta}{2}}$$



→ tražimo prosječnu rel. brzinu 1. čestice s svim ostalim

→ vjerojatnost da č. 2. zatvara ϑ sa 1.

$$dW = \frac{2\pi\sin\vartheta\, d\vartheta}{4\pi}$$

$$\int_0^\pi 2\pi\sin\vartheta\, d\vartheta = 2\pi\,(\cos\vartheta)\Big|_0^\pi = 2\pi\,(1+1) = 4\pi$$

$$\overline{\sin\frac{\vartheta}{2}} = \int_0^\pi \sin\frac{\vartheta}{2}\cdot\frac{\sin\vartheta}{2}\,d\vartheta = 2\int_0^\pi \sin^2\frac{\vartheta}{2}\cos\frac{\vartheta}{2}\,\underbrace{d\left(\frac{\vartheta}{2}\right)}_{d(\sin\frac{\vartheta}{2})} = \frac{2}{3}\sin^3\frac{\vartheta}{2}\Big|_0^\pi$$

$$= \frac{2}{3}$$

$$\Rightarrow \quad \underline{\bar{v}_r = \frac{4}{3}v}$$

Dz. 1.) Naći najvjerojatniji iznos brzine za Maxw. raspodjelu

✓ $v_w = ?$ $\sqrt{\frac{2xT}{m}}$

$$dW = \rho(v)\,d^3v = \left(\frac{m}{2\pi xT}\right)^{3/2} e^{-\frac{mv^2}{2xT}}\,d^3\vec{v} \qquad \vec{v},\ \vec{v}+d^3\vec{v}$$

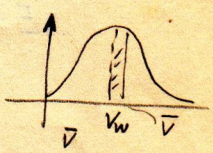
2.) $\bar{v} = ?$ $\sqrt{\frac{8xT}{\pi m}}$ ✓

$$dW_v = P(v) = 4\pi\left(\frac{m}{2\pi xT}\right)^{3/2} e^{-\frac{mv^2}{2xT}}\cdot v^2\,dv \qquad v,\ v+dv$$

$$\int_0^\infty dW_v = 1$$

✓ 3.) Naći broj čestica između srednje $\bar{v}$ & v_w brzine

✓ → naći $\Delta v = (\overline{v^2} - \bar{v}^2)^{1/2}$ neodređ. u brzini $\quad \overline{v^2} = \frac{3xT}{m}$



$$\int_{v_w}^{\bar{v}} dW_v = \text{dio od } 1$$

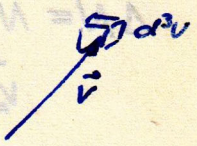
? 4.) $\bar{v}_{rel}$ za Maxw. raspodjelu

✓

Maxwell-ova raspodjela

$$dW = g(v)\,d^3v = f(\vec{v})\,d^3\vec{v} = \left(\frac{m}{2\pi\varkappa T}\right)^{3/2} e^{-\frac{mv^2}{2\varkappa T}}\,d^3\vec{v}$$

- s kojom vjeroj. nađemo česticu s brzinom $\vec{v}$ u d^3v



$$\int_{\mathbb{R}^3} f(\vec{v})\,d^3\vec{v} = 1$$

→ vjerojatnost po broju čestica

$$\underline{dn(\vec{v}) = N f(\vec{v})\,d^3v}$$

– broj čestica koje imaju brzinu $\vec{v}$

$$d\Omega = \frac{r^2 \sin\vartheta\, d\vartheta\, d\varphi}{r^2} = \sin\theta\, d\vartheta\, d\varphi$$

→ tip raspodjele za iznos brzine

$$d^3\vec{v} = dv_x\, dv_y\, dv_z = v^2 \sin\theta\, dv\, d\theta\, d\varphi = v^2\, d\Omega\, dv$$

$$\underbrace{F(v)}_{P(v)}\,dv = \int_\Omega f(\vec{v})\,d\Omega\, v^2 dv = 4\pi \left(\frac{m}{2\pi\varkappa T}\right)^{3/2} e^{-\frac{mv^2}{2\varkappa T}}\, v^2 dv$$

$$0 = F'(v) = 4\pi \left(\frac{m}{2\pi\varkappa T}\right)^{3/2} e^{-\frac{mv^2}{2\varkappa T}} \left[2v - v^2 \cdot \frac{mv}{\varkappa T}\right]$$

$$\Rightarrow \underline{v_w = \sqrt{\frac{2\varkappa T}{m}}}$$

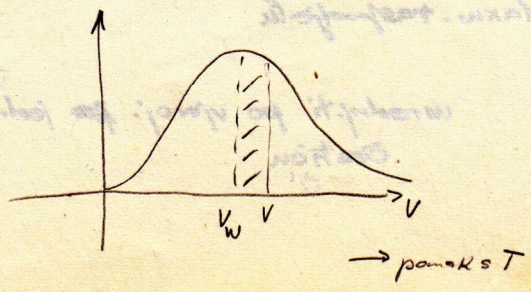
$$\underline{\bar{v}} = 4\pi \left(\frac{m}{2\pi\varkappa T}\right)^{3/2} \int_0^\infty e^{-\frac{mv^2}{2\varkappa T}}\, v^3 dv = \underline{\sqrt{\frac{8\varkappa T}{\pi m}}}$$

$$\frac{m\overline{v^2}}{2} = \bar{E} = \frac{3}{2}\varkappa T \quad \Rightarrow \quad \underline{\overline{v^2} = \frac{3\varkappa T}{m}}$$

disperzija

$$\Delta v = v - \bar{v} \quad \Rightarrow \quad \overline{\Delta v^2} = \underline{\Delta v = \sqrt{\overline{v^2} - \bar{v}^2} = \sqrt{\frac{\varkappa T}{m}\left(3 - \frac{8}{\pi}\right)}}$$

$$\frac{\Delta v}{\bar{v}} = \sqrt{\frac{3 - \frac{8}{\pi}}{\frac{8}{\pi}}} = \sqrt{\frac{3\pi - 8}{8}} = \sqrt{\frac{3\pi}{8} - 1}$$



3) Broj čestica s brzinom između V_w & $\bar{V}$

$$dn(\vec{v}) = Nf(\vec{v})\,d^3\vec{v} \Rightarrow dn(v) = N F(v)\,dv$$

$$\Delta N = N\int_{V_w}^{\bar{V}} F(v)\,dv = N\cdot 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\left[\int_0^{\sqrt{\frac{8kT}{\pi m}}} e^{-\frac{mv^2}{2kT}}\cdot v^2 dv - \int_0^{\sqrt{\frac{2kT}{m}}} e^{-\frac{mv^2}{2kT}}\cdot v^2 dv\right.$$

integrali izračunljivi samo $\int_0^\infty$ $\Rightarrow$ fja. pogreške koja je tabelirana

$$x = \sqrt{\frac{m}{kT}}\cdot v \qquad \text{bezdim. varijabla}$$

$$\Delta N = N\cdot 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\left(\frac{kT}{m}\right)^{3/2}\left[\int_0^{\sqrt{\frac{8}{\pi}}} e^{-\frac{x^2}{2}}\cdot x\,d\left(\frac{x^2}{2}\right) - \int_0^{\sqrt{2}} e^{-\frac{x^2}{2}}\,x\,d\left(\frac{x^2}{2}\right)\right] =$$

$N\cdot 4\cdot\pi\cdot\pi^{-3/2}\cdot 2^{-3/2}$

$N\cdot 2\cdot 2^{-1/2}\cdot\pi^{-1/2}$

$$= N\cdot\frac{2}{\sqrt{2\pi}}\left[\left(-xe^{-\frac{x^2}{2}}\right)_0^{\sqrt{8/\pi}} + \left(xe^{-\frac{x^2}{2}}\right)_0^{\sqrt{2}} + \int_0^{\sqrt{8/\pi}} e^{-\frac{x^2}{2}}dx - \int_0^{\sqrt{2}} e^{-\frac{x^2}{2}}dx\right] =$$

$$= N\cdot\frac{2}{\sqrt{2\pi}}\left[\sqrt{2}\,e^{-1} - \sqrt{\frac{8}{\pi}}\,e^{-\frac{4}{\pi}} + \int_0^{\sqrt{8/\pi}} e^{-\frac{x^2}{2}}dx - \int_0^{\sqrt{2}} e^{-\frac{x^2}{2}}dx\right] =$$

$$= N\cdot\left[\frac{2}{\sqrt{\pi}}e^{-1} - \frac{4}{\pi}e^{-\frac{4}{\pi}} + \sqrt{\frac{2}{\pi}}\int_0^{\sqrt{8/\pi}} e^{-\frac{x^2}{2}}dx - \sqrt{\frac{2}{\pi}}\int_0^{\sqrt{2}} e^{-\frac{x^2}{2}}dx\right]$$

$$\Rightarrow \underline{\frac{\Delta N}{N}} = \left[\frac{2}{\sqrt{\pi}}\cdot\frac{1}{e} - \frac{4}{\pi}e^{-\frac{4}{\pi}} + \underbrace{\phi\left(\sqrt{\frac{8}{\pi}}\right) - \phi\left(\sqrt{2}\right)}_{\approx 0{,}05}\right] \approx 0{,}13 \underline{= 13\,\%}$$

gdje $\underline{\phi(y) = \sqrt{\frac{2}{\pi}}\int_0^y e^{-\frac{x^2}{2}}dx}$ Bronštajn 88 str.

4) Srednja rel. brzina za Maxw. raspodjelu

$$\bar{V}_r = \overline{|\vec{V} - \vec{V}'|}$$

usrednjiti po vjerojatnosti $\vec{V}$ jedne, a zatim za drugu česticu $\vec{V}'$

$$dW_{12} = dW_1 \cdot dW_2 = f(\vec{v})\,d^3\vec{v} \cdot f(\vec{v}')\cdot d^3\vec{v}'$$

$m_1 = m_2 = m$

vjerojatnost da istovremeno 1. čestica ima brzinu $\vec{v}, \vec{v}+d^3\vec{v}$, a 2. čestica $\vec{v}', \vec{v}'+d^3\vec{v}'$

$$dW_{12} = \left(\frac{m}{2\pi\varkappa T}\right)^3 e^{-\frac{m}{2\varkappa T}(v^2+v'^2)}\, d^3\vec{v}\, d^3\vec{v}'$$

$$\overline{V_r} = \overline{|\vec{v}-\vec{v}'|}$$

$$\overline{V_r} = \left(\frac{m}{2\pi\varkappa T}\right)^3 \int_{v+v'}^{|\vec{v}-\vec{v}'|} \hat{V}_r\, e^{-\frac{m}{2\varkappa T}(v^2+v'^2)}\, d^3\vec{v}\, d^3\vec{v}'$$

→ nove varijable: $\vec{V} = \dfrac{\vec{v}+\vec{v}'}{2}$ $\qquad \vec{V}_r = \vec{v}-\vec{v}'$

$V_x = \dfrac{v_x+v_x'}{2}$ $\qquad V_{r_x} = v_x - v_x'$

$\vdots$ $\qquad V_{r_y} = v_y - v_y'$

$$d^3\vec{v}\, d^3\vec{v}' = J\, d^3\vec{V}\, d^3\vec{V}_r$$

$$J = \frac{\partial(V_x, V_y, V_z, V_x', V_y', V_z')}{\partial(V_x, V_y, V_z, V_{rx}, V_{ry}, V_{rz})} = \frac{1}{\dfrac{\partial(V_x, V_y, V_z, V_{rx}, V_{ry}, V_{rz})}{\partial(V_x, V_y, V_z, V_x', V_y', V_z')}} = 1$$

$$\Rightarrow J^{-1} = \begin{vmatrix} \frac{\partial V_x}{\partial V_x} & \frac{\partial V_y}{\partial V_x} & \frac{\partial V_z}{\partial V_x} & \frac{\partial V_{rx}}{\partial V_x} & \frac{\partial V_{ry}}{\partial V_x} & \frac{\partial V_{rz}}{\partial V_x} \\ & & & & & \\ & & & & & \end{vmatrix} = \begin{vmatrix} \frac12 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac12 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac12 & 0 & 0 & 1 \\ \frac12 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac12 & 0 & 0 & -1 & 0 \\ 0 & 0 & \frac12 & 0 & 0 & -1 \end{vmatrix} =$$

$$= \frac12 \begin{vmatrix} \frac12 & 0 & 0 & 1 & 0 \\ 0 & \frac12 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ \frac12 & 0 & 0 & -1 & 0 \\ 0 & \frac12 & 0 & 0 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & \frac12 & 0 & 1 & 0 \\ 0 & 0 & \frac12 & 0 & 1 \\ \frac12 & 0 & 0 & 0 & 0 \\ 0 & \frac12 & 0 & -1 & 0 \\ 0 & 0 & \frac12 & 0 & -1 \end{vmatrix} = \frac14 \begin{vmatrix} \frac12 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac12 & 0 & 0 & -1 \end{vmatrix} - \frac12 \begin{vmatrix} 0 & \frac12 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \frac12 & 0 & 0 & 0 \\ 0 & \frac12 & 0 & -1 \end{vmatrix} + \frac12 \begin{vmatrix} 0 & \frac12 & 0 & 1 \\ \frac12 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac12 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & \frac12 & 1 \\ \frac12 & 0 & 0 & 0 \\ 0 & \frac12 & 0 & 0 \\ 0 & 0 & \frac12 & -1 \end{vmatrix}$$

$$= \frac18 \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} - \frac14 \begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ \frac12 & 0 & 0 \end{vmatrix} + \frac14 \begin{vmatrix} 0 & -1 & 0 \\ \frac12 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} + \frac12 \begin{vmatrix} 0 & 0 & -1 \\ \frac12 & 0 & 0 \\ 0 & \frac12 & 0 \end{vmatrix} - \frac14 \begin{vmatrix} \frac12 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} - \frac12 \begin{vmatrix} \frac12 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & \frac12 & 0 \end{vmatrix} + \frac12 \begin{vmatrix} \frac12 & 0 & 0 \\ 0 & \frac12 & 0 \\ 0 & 0 & -1 \end{vmatrix} - \begin{vmatrix} \frac12 & 0 & 0 \\ 0 & \frac12 & 0 \\ 0 & 0 & \frac12 \end{vmatrix}$$

$$= -\frac18 + \frac18 - \frac18 + \frac18 - \frac18 - \frac18 - \frac18 - \frac18$$

$$\Rightarrow d^3\vec{V}\, d^3\vec{V}' = d^3\vec{V} \cdot d^3\vec{V}_r$$

$$V^2 + V'^2 = \frac{(\vec{V}+\vec{V}')^2 + (\vec{V}-\vec{V}')^2}{2} = 2V^2 + \frac{V_r^2}{2}$$

$$\Rightarrow \overline{V_r} = \left(\frac{m}{2\pi\varkappa T}\right)^3 \underbrace{\int e^{-\frac{mV^2}{\varkappa T}} d^3\vec{V}}_{4\pi\int_0^\infty V^2 e^{-\frac{mV^2}{\varkappa T}} dV} \cdot \underbrace{\int V_r e^{-\frac{mV_r^2}{4\varkappa T}} d^3\vec{V}_r}_{4\pi\int_0^\infty V_r^3 e^{-\frac{mV_r^2}{4\varkappa T}} dV}$$

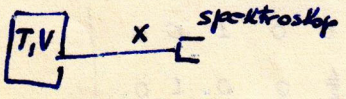
$$\underline{\overline{V_r} = 4\sqrt{\frac{\varkappa T}{\pi m}} = \sqrt{2}\cdot\overline{V}}$$

→ Boltzman proširio Maxw. rasp. $\rho(E) \sim e^{-\frac{E}{\varkappa T}}$

Dz.

① zvijezda - užareni plin (molek. različ. brzine) - podvrgnut Boltzm. (nema pos. interakcije a ne mora biti 1-A)

⇒ možemo operirati s μ-prost.



iz širine spektr. linija ⇒ temp.

U zatvorenom volumenu kod temp. T nalazi se plin atoma mase m. Atomi emitiraju svjetlost koja prolazi kroz otvor u smjeru x i promatra se u spektroskopu. Atom u mirovanju emitira svjetlost frekvencije ω_0, međutim ako se atom giba, tada se zbog Doppler efekta opaža svjetlost frekv.

$$\omega \simeq \omega_0\left(1 + \frac{V_x}{c}\right) \quad \text{aprox. relativist. rel.}$$

Pretpostaviti da se čestice podvrgavaju Maxwell. raspodjeli i izraziti srednju frekvenciju $\underline{\overline{\omega}}$ i disperziju $\underline{\Delta\omega}$

pona

$$\frac{V(mxT)^{3/2}}{h^3 N} \gg 1$$

→ ponašanje sistema nije odred. samo s E_k , već i sa E_p

Boltzmann-ova raspodjela

$$dW = f(\vec{r},\vec{p})\, d^3\vec{p}\, d^3\vec{r} = \frac{\frac{1}{h^3} e^{-\frac{E(\vec{r},\vec{p})}{xT}}}{\underbrace{\frac{1}{h^3}\int_\mu d^3\vec{p}\, d^3\vec{r}\, e^{-\frac{E(\vec{r},\vec{p})}{xT}}}_{Z}}$$

Z - partic. fja. 1 mol. kl.

E - ukupna energija (čestice-sistema)

po μ-prostoru (ali idealan plin nema interakcija)

ali moguće djel. vanj. sila

Boltz. → Maxw za 1A plin (idealni) & bez vanj. polja

$$\bar{E} = -\left(\frac{\partial}{\partial\beta}\ln Z\right)_V \qquad \beta = \frac{1}{xT}$$

② Izračunati spec. toplinu čestice, ako je njena potenc. energija

$$U(x,y,z) = Ar^6$$

→ formalizam dozvoljava i tretiranje i jednog polimer. objekta

$$E = H = \frac{p^2}{2m} + Ar^6$$

$$\bar{E} = \int_\mu E\, dW = \frac{\int_\mu \frac{p^2}{2m} e^{-\frac{\frac{p^2}{2m}+Ar^6}{xT}} d^3\vec{r}\, d^3\vec{p}}{\int_\mu d^3\vec{p}\, d^3\vec{r}\, e^{-\frac{\frac{p^2}{2m}+Ar^6}{xT}}} + \frac{\int_\mu Ar^6 e^{-\frac{\frac{p^2}{2m}+Ar^6}{xT}} d^3\vec{r}\, d^3\vec{p}}{\int_\mu d^3\vec{p}\, d^3\vec{r}\, e^{-\frac{\frac{p^2}{2m}+Ar^6}{xT}}}$$

Maxw. resp.

$$\bar{H} = \frac{3}{2}xT + \frac{\int_\mu d^3\vec{r}\, Ar^6 e^{-\beta Ar^6}}{\int_\mu d^3\vec{r}\, e^{-\beta Ar^6}} \qquad \beta = \frac{1}{xT}$$

$$= \frac{3}{2}xT - \frac{\partial}{\partial\beta}\ln \underbrace{\frac{1}{h^3}\int d^3\vec{r}\, e^{-\beta Ar^6}}_{Z}$$

$$Z = \int_\mu d^3\vec{r}\, e^{-\beta Ar^6}$$

uvodimo $\beta A r^6 = t$ $\quad r = \left(\frac{t}{\beta A}\right)^{1/6}$

$$dr = \frac{1}{6}\cdot\left(\frac{1}{\beta A}\right)^{1/6} t^{-\frac{5}{6}} dt$$

$$Z = 4\pi\int_0^\infty dr\, r^2 e^{-\beta A r^6} = 4\pi\left(\frac{1}{\beta A}\right)^{1/3}\cdot\left(\frac{1}{\beta A}\right)^{1/6}\cdot\int_0^\infty e^{-t} t^{\frac{2}{6}}\cdot t^{-\frac{5}{6}}\, dt$$

$$\int_0^\infty e^{-t} t^{-\frac{1}{2}} dt = \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$Z = \left(\frac{1}{\beta}\right)^{1/2}$$

$$\ln \beta^{-1/2}$$

$$\bar{H} = \frac{3}{2}xT - \frac{\partial}{\partial\beta}\ln\left(\frac{1}{\beta}\right)^{1/2} = \frac{3}{2}xT - \beta^{1/2}\cdot\left(-\frac{1}{2}\beta^{-\frac{3}{2}}\right) = \frac{3}{2}xT + \frac{1}{2}\beta^{-1}$$

$$\bar{H} = \frac{3}{2}xT + \frac{1}{2}xT = 2xT \quad \Rightarrow \quad C_{V_{1\,mol.}} = \left(\frac{\partial\bar{H}}{\partial T}\right)_V = 2x$$

II z.

③ Za harm. oscilator (trodim.) $H = \frac{p^2}{2m} + \frac{m\omega^2}{2} r^2$ izotropni (tj: s istim ω u svim smjerovima)

nađi $\overline{M_z}$ (z-komp. $\vec{M} = \vec{r} \times \vec{p}$ $M_z = xp_y - yp_x$)

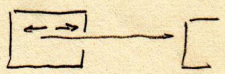
$$\overline{M^2} = \overline{M_x^2 + M_y^2 + M_z^2}$$

Rješenje:

① str. 11.

vjerojat. da molek. ima komp. V_x → Maxw. raspo.

$$f(V_x)\,dV_x = \left(\frac{m}{2\pi\varkappa T}\right)^{1/2} e^{-\frac{mV_x^2}{2\varkappa T}}\,dV_x$$



$$\overline{\omega} = \omega_0\left(1 + \frac{\overline{V_x}}{c}\right) = \omega_0$$

jer $\overline{V_x}$ nep. fja.

$$\overline{V_x} = \left(\frac{m}{2\pi\varkappa T}\right)^{1/2} \int_{-\infty}^{\infty} V_x\, e^{-\frac{mV_x^2}{2\varkappa T}}\,dV_x = 0$$

$$\overline{\omega^2} = \omega_0^2\left(1 + \frac{2\overline{V_x}}{c} + \frac{\overline{V_x^2}}{c^2}\right) = \omega_0^2\left(1 + \frac{\overline{V_x^2}}{c^2}\right)$$

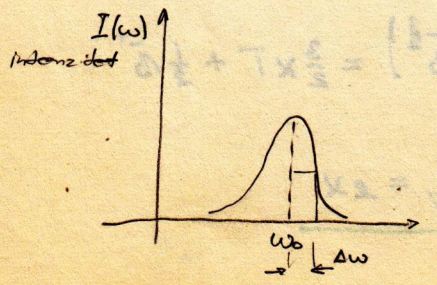
$$\overline{V_x^2} = \left(\frac{m}{2\pi\varkappa T}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{mV_x^2}{2\varkappa T}} V_x^2\,dV_x = \frac{\varkappa T}{m}$$

jer $\frac{m\overline{V_x^2}}{2} = \frac{1}{2}\varkappa T$

$$\Rightarrow \overline{\omega_0^2} = \omega_0^2\left(1 + \frac{1}{c^2}\frac{\varkappa T}{m}\right)$$

$$\Rightarrow \overline{\omega^2} - \overline{\omega}^2 = \omega_0^2\left(1 + \frac{1}{c^2}\frac{\varkappa T}{m} - 1\right)$$

$$\Rightarrow \Delta\omega = \frac{\omega_0}{c}\sqrt{\frac{\varkappa T}{m}}$$



spektri zvijezda: kad bi svi miravali bila bi oštra linija

disperzija dolazi zbog Doppl. efekta

→ iz impar. šir. spektra omog. nađi T zvijezde

Rješ.

③ na 12.

Jednim. harm. oscilator - izotropn.: $\omega_x = \omega_y = \omega_z = \omega$

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} r^2$$

$$\vec{M} = \vec{r} \times \vec{p} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

→ nema interakcije tih čestica - oscilatora

$$M_z = x p_y - y p_x$$

$$\overline{M_z} = \frac{\int_\mu (x p_y - y p_x)\, e^{-\beta\left(\frac{p^2}{2m} + \frac{m\omega^2}{2} r^2\right)} d^3\vec{r}\, d^3\vec{p}}{\int_\mu e^{-\beta\left(\frac{p^2}{2m} + \frac{m\omega^2}{2} r^2\right)} d^3\vec{r}\, d^3\vec{p}}$$

$$= \frac{\int_{-\infty}^{\infty} x e^{-\beta \frac{m\omega^2}{2} x^2} dx \cdot \int_{-\infty}^{\infty} p_y e^{-\beta \frac{p_y^2}{2m}} dp_y}{\int_{-\infty}^{\infty} e^{-\beta \frac{m\omega^2}{2} x^2} dx \cdot \int_{-\infty}^{\infty} e^{-\beta \frac{p_y^2}{2m}} dp_y} - \frac{\int_{-\infty}^{\infty} y e^{-\beta \frac{m\omega^2}{2} y^2} dy \cdot \int_{-\infty}^{\infty} p_x e^{-\beta \frac{p_x^2}{2m}} dp_x}{\int_{-\infty}^{\infty} e^{-\beta \frac{m\omega^2}{2} y^2} dy \cdot \int_{-\infty}^{\infty} e^{-\beta \frac{p_x^2}{2m}} dp_x} = 0$$

$$\overline{M^2} = \overline{(M_x^2 + M_y^2 + M_z^2)} = 3\overline{M_z^2}$$

jer su x i p_x nezavisne varijable a osim toga isprozl. iz Boltz. resp.

$$= 3\overline{(x^2 p_y^2 + y^2 p_x^2 - 2xy\, p_x p_y)} = 3(\overline{x^2}\,\overline{p_y^2} + \overline{y^2}\cdot\overline{p_x^2} - 2\overline{x}\cdot\overline{y}\cdot\overline{p_x}\cdot\overline{p_y})$$

$$= 6\,\overline{x^2}\cdot\overline{p_y^2}$$

$$\overline{x^2} = \frac{\int_{-\infty}^{\infty} x^2 e^{-\beta \frac{m\omega^2}{2} x^2} dx}{\int_{-\infty}^{\infty} e^{-\beta \frac{m\omega^2}{2} x^2} dx} \qquad \overline{p_y^2} = \frac{\int_{-\infty}^{\infty} p_y^2 e^{-\beta \frac{p_y^2}{2m}} dp_y}{\int_{-\infty}^{\infty} e^{-\beta \frac{p_y^2}{2m}} dp_y}$$

$$\overline{x^2} = -\frac{2}{m\omega^2} \frac{\partial}{\partial \beta} \ln \int_{-\infty}^{\infty} e^{-\beta \frac{m\omega^2}{2} x^2} dx \qquad \overline{p_y^2} = -2m \frac{\partial}{\partial \beta} \ln \int_{-\infty}^{\infty} e^{-\beta \frac{p_y^2}{2m}} dp_y$$

$$t^2 = \beta \frac{m\omega^2}{2} x^2 \qquad t^2 = \beta \frac{1}{2m} p_y^2$$

$$x = \frac{1}{\omega}\sqrt{\frac{2}{\beta m}}\, t \qquad p_y = \sqrt{\frac{2m}{\beta}}\, t$$

$$\frac{1}{\omega}\sqrt{\frac{2}{\beta m}} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{1}{\omega}\sqrt{\frac{2\pi}{\beta m}} = \sqrt{\frac{2\pi}{m\omega^2}}\, \beta^{-1/2} \qquad \sqrt{\frac{2m}{\beta}} \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\frac{2m\pi}{\beta}} = \sqrt{2m\pi}\cdot \beta^{-1/2}$$

$$2\int_0^{\infty} e^{-t^2} dt \qquad 2\int_0^{\infty} e^{-t^2} dt = 2\cdot\frac{\sqrt{\pi}}{2}$$

$$\frac{\partial}{\partial x} \ln f(x) B(y) = \frac{\partial}{\partial x} \ln f(x)$$

$$\overline{x^2} = -\frac{2}{m\omega^2}\frac{\partial}{\partial\beta}\ln\beta^{-1/2} \qquad \overline{p_y^2} = -2m\frac{\partial}{\partial\beta}\ln\beta^{-1/2}$$

$$\beta^{1/2} \quad -\frac{1}{2}\beta^{-3/2} \qquad \beta^{1/2}\left(-\frac{1}{2}\right)\beta^{-3/2}$$

$$\overline{x^2} = \frac{1}{m\omega^2}\,xT \qquad \overline{p_y^2} = mxT$$

$$\Rightarrow \quad \overline{M^2} = 6\cdot\overline{x^2}\,\overline{p_y^2} = 6\cdot\frac{1}{m\omega^2}xT\cdot mxT$$

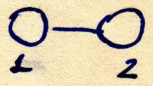
$$\underline{\overline{M^2} = 6\left(\frac{xT}{\omega}\right)^2}$$

IIz.

① Sistem se sastoji od n neinteragirajućih dvoatomskih molekula, a $\forall$ je opisana po Hamiltonijanu

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{\alpha}{2}|\vec{r}_1 - \vec{r}_2|^2$$

harm. oscilator



dva atoma kaj se drže harm. silama

Naći slob. energiju F
top. kapacitet C_V & $\overline{|\vec{r}_1 - \vec{r}_2|^2}$

fja. raspodjele (Maxwell - Boltzmann) (bez interakcije)

$$f(\vec{r},\vec{p}) = \frac{\frac{1}{h^{Ns}} e^{-\beta H(\vec{r},\vec{p})}}{Z} \qquad Z = \frac{1}{N!\,h^{Ns}} \int_{\Gamma} e^{-\beta H(\vec{r},\vec{p})} d\phi$$

N-čestica $\qquad d\phi = d^3\vec{r}_1 \dots d^3\vec{r}_N\, d^3\vec{p}_1 \dots d^3\vec{p}_N$

$$H(\vec{r},\vec{p}) = T(\vec{p}_1,\dots\vec{p}_N) + U(\vec{r}_1,\dots\vec{r}_N)$$

$$F = -\frac{1}{\beta}\ln Z = F(T,V)$$

$$U = -\frac{\partial}{\partial\beta}\ln Z = \frac{\partial}{\partial\beta}(\beta F) \quad \Rightarrow C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V \qquad P = -\left(\frac{\partial F}{\partial V}\right)_T$$

Rješ.

① sa B. N -dia A. mol.

sa $H_i = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{\alpha}{2}|\vec{r}_1-\vec{r}_2|^2$

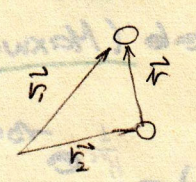
$$H = \sum_{i=1}^{N} H_i \qquad Z = \frac{1}{N!\,h^{Ns}} \int_{\Gamma} e^{-\beta\sum_{i=1}^{N} H_i} \prod_{i=1}^{N} d^3\vec{r}_i\, d^3\vec{p}_i$$

$$Z = \frac{1}{N!\,h^{Ns}}\left[\int_{\Gamma} e^{-\beta\left(\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}+\frac{\alpha}{2}|\vec{r}_1-\vec{r}_2|^2\right)} d^3\vec{p}_1 d^3\vec{p}_2\, d^3\vec{r}_1 d^3\vec{r}_2\right]^N$$

→ prom. gib. molek. kao ojačine i mj. uduž. atoma

C.M. sist.

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$



$$\vec{P} = (m_1+m_2)\dot{\vec{R}} \qquad \vec{p} = \underbrace{\frac{m_1 m_2}{m_1+m_2}}_{\mu}\dot{\vec{r}} \qquad M = m_1+m_2$$

$$\Rightarrow \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{P^2}{2M} + \frac{p^2}{2\mu}$$

$$d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{r}_1\, d^3\vec{r}_2 = \frac{\partial(\vec{r}_1,\vec{r}_2,\vec{p}_1,\vec{p}_2)}{\partial(\vec{R},\vec{r},\vec{P},\vec{p})} = \overbrace{\left|\frac{\partial(\vec{r}_1,\vec{r}_2)}{\partial(\vec{R},\vec{r})}\right|\left|\frac{\partial(\vec{p}_1,\vec{p}_2)}{\partial(\vec{P},\vec{p})}\right|}^{1} d^3\vec{R}\, d^3\vec{r}\, d^3\vec{P}\, d^3\vec{p}$$

$$= d^3\vec{R}\, d^3\vec{r}\, d^3\vec{P}\, d^3\vec{p}$$

$$Z = \frac{1}{N!\,h^{N\cdot s}}\left[\int_{V'} e^{-\beta\left(\frac{P^2}{2M} + \frac{p^2}{2\mu} + \frac{\alpha}{2}r^2\right)} d^3\vec{P}\, d^3\vec{p}\, d^3\vec{R}\, d^3\vec{r}\right]^N$$

H neovisi o $\vec{R}$ $\rightarrow V$

po kutev. $\rightarrow 4\pi$

$$Z = \frac{1}{N!\,h^{N\cdot s}}\left(V\cdot(4\pi)^3\right)^N \left(\int_0^\infty e^{-\beta\frac{P^2}{2M}} P^2 dP\right)^N \left(\int_0^\infty e^{-\beta\frac{p^2}{2\mu}} p^2 dp\right)^N \left(\int_0^\infty e^{-\beta\frac{\alpha}{2}r^2} r^2 dr\right)^N$$

$x = \sqrt{\frac{\beta}{2M}}\,P$

$x^2 = \frac{\beta}{2M}P^2 \qquad x^2 = \frac{\beta}{2\mu}p^2 \qquad x^2 = \frac{\beta\alpha}{2}r^2$

$dP = \sqrt{\frac{2M}{\beta}}\,dx \qquad dp = \sqrt{\frac{2\mu}{\beta}}\,dx \qquad dr = \sqrt{\frac{2}{\beta\alpha}}\,dx$

$\int_0^\infty e^{-x^2}\cdot x^2 dx$

$\frac{2M}{\beta}\cdot\sqrt{\frac{2M}{\beta}}\int_0^\infty e^{-x^2}x^2 dx$

$$Z = \frac{1}{N!\,h^{N\cdot s}}\left(V\cdot(4\pi)^3\right)^N\cdot\left(\frac{2M}{\beta}\right)^{\frac{3N}{2}}\cdot\left(\frac{2\mu}{\beta}\right)^{\frac{3N}{2}}\cdot\left(\frac{2}{\beta\alpha}\right)^{\frac{3N}{2}}\cdot\left(\int_0^\infty e^{-x^2}x^2 dx\right)^{3N}$$

$= \frac{1}{2}\cdot\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$ $\qquad \frac{\pi^{\frac{3N}{2}}}{4^{3N}}$

$(4\pi)^{3N}\, 2^{\frac{3N}{2}}\cdot\frac{\pi^{\frac{3N}{2}}}{2^{6N}} = \pi^{\frac{9N}{2}}\cdot 2^{\frac{9N}{2}}$

$$Z = \frac{1}{N!\,h^{N\cdot s}}\, V^N\cdot(2\pi\times T)^{\frac{9N}{2}}\cdot\left(\frac{M\mu}{\alpha}\right)^{\frac{3N}{2}}$$

1) $F = -\varkappa T \ln Z = -N\varkappa T \ln\left[\frac{V}{h^{ns}}(2\pi\varkappa T)^{\frac{9}{2}}\left(\frac{M\mu}{\alpha}\right)^{\frac{3}{2}}\right]$

2) $U = -\frac{\partial}{\partial\beta}\ln Z = -\frac{\partial}{\partial\beta}\ln\beta^{-\frac{9N}{2}} = -\beta^{\frac{9N}{2}}\cdot\left(-\frac{9N}{2}\right)\beta^{-\frac{9N}{2}-1} = \frac{9N}{2}\beta^{-1}$

$$U = \frac{9}{2}\frac{N}{\beta} = \frac{9}{2}N\varkappa T$$

3) $C_v = \frac{9}{2}N\varkappa$

$$\overline{r_i^2} = \frac{\int_\Gamma r_i^2 e^{-\beta\sum H_i}\,d\phi}{\int_\Gamma e^{-\beta\sum H_i}\,d\phi}$$

$$\overline{r^2} = \left(\frac{\int_0^\infty\int_0^\infty\int_0^\infty r^2 e^{-\beta\left[\frac{P^2}{2M}+\frac{p^2}{2\mu}+\frac{\alpha}{2}r^2\right]}P^2p^2r^2\,dP\,dp\,dr}{\int_0^\infty\int_0^\infty\int_0^\infty e^{-\beta\left[\frac{P^2}{2M}+\frac{p^2}{2\mu}+\frac{\alpha}{2}r^2\right]}P^2p^2r^2\,dP\,dp\,dr}\right) =$$

$$= \left(\frac{\int_0^\infty r^2 e^{-\frac{\beta\alpha}{2}r^2}r^2\,dr}{\int_0^\infty e^{-\frac{\beta\alpha}{2}r^2}r^2\,dr}\right) = -\frac{2}{\alpha}\frac{\partial}{\partial\beta}\ln\underbrace{\frac{1}{h^3}\int_0^\infty e^{-\frac{\beta\alpha}{2}r^2}r^2\,dr}_{Z}$$

$$Z = \left(\frac{2}{\beta\alpha}\right)^{3/2}\cdot\frac{\sqrt{\pi}}{4}$$

$\ln\beta^{-\frac{3}{2}} = -\frac{3}{2}\ln\beta$

$$= -\frac{2}{\alpha}\frac{\partial}{\partial\beta}\ln Z = -\frac{2}{\alpha}\cdot\left(-\frac{3}{2}\beta^{-1}\right)$$

$$= \frac{3}{\alpha}\varkappa T\cdot\cancel{N}$$ pogreška na vježb.

$$\frac{\alpha}{2}\overline{r^2} = \frac{\cancel{\alpha}}{2}\cdot\frac{3}{\cancel{\alpha}}\varkappa T = \frac{3}{2}\varkappa T$$

② Plin ~~je~~ sastavljen od ultrarelativističkih čestica N

$E = c\cdot p$ $\quad E \neq \frac{p^2}{2m}$ $\quad p = \hbar k$

Naći: F, U, S, P, C_v

$$Z = \frac{1}{N!\,h^{ns}}\int_\Gamma e^{-\beta H}\,d\phi \qquad H_i = cp_i + U_i \qquad U_i = \begin{cases}0 & \in V\\ \infty & \notin V\end{cases}$$

$$Z = \frac{1}{N!\,h^{ns}}\left(\int_\mu e^{-\beta c p}\,d^3\vec{r}\,d^3\vec{p}\right)^N$$

$p_x, p_y, p_z = p$ axes

$x = \beta c p \quad dp = \frac{1}{\beta c}dx$

$$Z = \frac{1}{N!\,h^{ns}}(V\cdot 4\pi)^N\int_0^\infty e^{-\beta c p}p^2\,dp = \frac{1}{N!\,h^{ns}}\left(\frac{V\cdot 4\pi}{c^3\beta^3}\right)^N\left(\int_0^\infty e^{-x}x^2\,dx\right)^N = \left(\frac{8\pi V}{\beta^3c^3}\right)^N$$

$\Gamma(3) = 2\cdot 1$

$$F = -\frac{1}{\beta}\ln Z = -N\varkappa T \ln \frac{8\pi V}{\beta^3 c^3}$$

$$U = -\frac{\partial}{\partial\beta}\ln Z = -\frac{\partial}{\partial\beta}\ln \beta^{-3N} = \frac{3N}{\beta} = 3N\varkappa T$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 3N\varkappa$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{\partial}{\partial T}\left(N\varkappa T \ln \frac{8\pi V}{c^3}\cdot \varkappa^3 T^3\right)$$

$$= N\varkappa \ln \frac{8\pi V}{c^3}\varkappa^3 T^3 + 3N\varkappa$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{N\varkappa T}{V} = \underline{\underline{\frac{1}{3}\frac{U}{V}}}$$

gustoća energije

Kod zrač. crnog tijela

→ foton $E = c\cdot p$

Dz.

① Sistem od jedne čestice je opisan sa $H = \frac{p^2}{2m} + ax^2 + bx$

Za N čestica naći:

$$F, S, U, C_V$$

② Sistem se sastoji od N molekula od kojih svaka ima el. dipolni moment $\vec{p}$. Sistem je stavljen u konst. vanj. el. polje. Izračunati polarizaciju sistema P i pokazati da u slučaju ako je $p\cdot E \ll \varkappa T$ da je dielektr. konstanta takvog plina data sa

$$\varepsilon = 1 + 4\pi\frac{N}{V}\frac{p^{3\,2}}{3\varkappa T}$$

tj. treba izrač. sr. pot. energ. sistema

$$U = -\vec{p}\cdot\vec{E}$$

→ sr. dip. moment.

polariz. u s-jem $\vec{E}$ Rješ:

$$P = \frac{N}{V}p\left\{\coth\frac{pE}{\varkappa T} - \frac{\varkappa T}{pE}\right\}$$

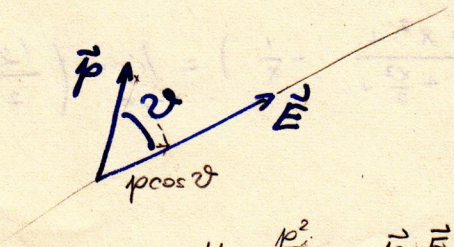
Napom.: $Z = A \int e^{-\beta H(\vec{q},\vec{p})} d\phi$

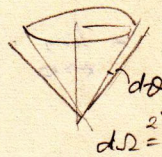
zbog nerazl. lokaliz. čest. unutar faznog volum.

Klas. stat. fizika $A = \frac{1}{h^{3N}}$

Rpos.

② na 15. Dipol. u el. polju





$d\Omega = \frac{2\pi r \sin\vartheta\, r\, d\vartheta}{r^2} = 2\pi \sin\vartheta\, d\vartheta$

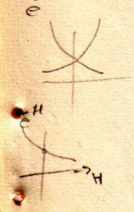
$U = -\vec{p}\cdot\vec{E}$ za 1 molek

$H_i = \frac{p_i^2}{2m_i} - \vec{p}_i\cdot\vec{E}$

$\varrho = \frac{e^{-\beta\sum_{i=1}^{N} H_i}}{\int_\Gamma e^{-\beta\sum_{i=1}^{N} H_i} d\phi}$

fja. raspodjele $f = \frac{e^{\beta\vec{p}\vec{E}} \sin\vartheta}{\int e^{\beta\vec{p}\vec{E}} \sin\vartheta\, d\vartheta}$

$\overline{\cos\vartheta_i} = \frac{\int_\Gamma \cos\vartheta_i\, e^{-\beta\sum_{i=1}^{N} H_i}}{\int_\Gamma e^{-\beta\sum_{i=1}^{N} H_i} d\phi} = \frac{\int \cos\vartheta_i\, e^{-\beta(-pE\cos\vartheta_i)}}{\int e^{\beta(-pE\cos\vartheta_i)}}$



$dW' \sim e^{-\beta(-p\cdot E)} \cdot d\Omega$

$dW' = A\, e^{\beta pE\cos\vartheta}\, 2\pi \sin\vartheta\, d\vartheta$ $\quad \int_0^\pi dW' = 1 \Rightarrow A = \frac{1}{2\pi \int_0^\pi e^{\beta pE\cos\vartheta} \sin\vartheta\, d\vartheta}$

polariz = dipolni mom. jed. volum.

$P = \frac{N\cdot p\langle\cos\vartheta\rangle}{V}$ $\quad \langle\cos\vartheta\rangle = \frac{\int_0^\pi \cos\vartheta\, e^{\frac{pE}{kT}\cos\vartheta} \sin\vartheta\, d\vartheta}{\int_0^\pi e^{\frac{pE}{kT}\cos\vartheta} \sin\vartheta\, d\vartheta}$

$x = \frac{pE}{kT}$ $\quad \langle\cos\vartheta\rangle = \frac{\partial}{\partial x} \ln \int_0^\pi e^{x\cos\vartheta} \sin\vartheta\, d\vartheta$

$y = \cos\vartheta$, $dy = -\sin\vartheta\, d\vartheta$

$\int_0^\pi e^{x\cos\vartheta} \sin\vartheta\, d\vartheta = -\int_1^{-1} e^{x\cdot y} dy = \frac{1}{x}\int_{-1}^{1} e^{x\cdot y} dy\cdot x = \frac{1}{x}\cdot e^{xy}\Big|_{-1}^{1} = \frac{1}{x}(e^x - e^{-x}) = \frac{2\,\mathrm{sh}\,x}{x}$

$\Rightarrow \langle\cos\vartheta\rangle = \frac{x}{e^x - e^{-x}}\left[-\frac{1}{x^2}(e^x - e^{-x}) + \frac{1}{x}(e^x + e^{-x})\right] = -\frac{1}{x} + \frac{e^x + e^{-x}}{e^x - e^{-x}} = -\frac{1}{x} + \mathrm{cth}\,x$

$= -\frac{1}{x} + \mathrm{cth}\,x$

$\Rightarrow P = \frac{N p}{V}\cdot\left\{\mathrm{cth}\frac{pE}{kT} - \frac{kT}{pE}\right\}$ $\quad L\left(\frac{pE}{kT}\right)$ - Langeveno fja

Razvoj : $\frac{pE}{\varkappa T} \ll 1 \quad \Rightarrow x \ll 1$

$$\langle \cos\vartheta \rangle = -\frac{1}{x} + \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{45} - \frac{x^7}{4725} + \dots \approx \underline{\underline{\frac{x}{3}}}$$

$$\lim_{x\to 0} \left(\operatorname{cth} x - \frac{1}{x}\right) = \lim_{x\to 0} \left(\frac{1 + \cancel{x} + \frac{x^2}{2} + \dots \quad 1 - \cancel{x} + \frac{x^2}{2} + \dots}{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right) - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right)} - \frac{1}{x} \right)$$

$$= \lim_{x\to 0} \left(\frac{2 + x^2 + \dots}{2x + \frac{x^3}{3} + \dots} - \frac{1}{x} \right) = \lim_{x\to 0} \left(\frac{(2 + x^2)}{2x\left(1 + \frac{x^2}{6}\right)} - \frac{1}{x} \right) \qquad \frac{1}{1 + \frac{x^2}{6}} \approx 1 - \frac{x^2}{6} + \dots$$

$$= \frac{1}{3}$$

$$\underline{P = \frac{Np}{V} \cdot \frac{pE}{3\varkappa T}}$$ Curieev-zakon za $\varkappa T \gg pE$

$$D = \varepsilon E = E + 4\pi P$$

$$\Rightarrow P = \frac{\varepsilon - 1}{4\pi} E \qquad \frac{\varepsilon - 1}{4\pi} = \frac{N}{V} \cdot \frac{1}{3} \frac{p^2}{\varkappa T}$$

$$\Rightarrow \underline{\varepsilon = 1 + \frac{4\pi}{3} \cdot \frac{N}{V} \cdot \frac{p^2}{\varkappa T}}$$

– analogno i za magnetiz. $\vec{m}, \vec{H} \Rightarrow U = -\vec{m} \cdot \vec{H}$

Iz.

① Na koju temp. treba zagrijati plin $T_1 = 100\,K$ ako je potenc. energ. čestica

$$U = -a\cos\varphi \qquad 0 \le \varphi \le \pi$$

$$(a = \vec{E} \cdot \vec{d}) \qquad a = 5{,}52 \cdot 10^{-16}\,erg$$

da bi se srednja energ. molekula povećala za

$$\Delta \bar{E} = 3{,}68 \cdot 10^{-18}\,erg.$$

1° : srednja energ. T_1

2°

R: $T_2 = 2T_1$

$$\bar{E} = \frac{3}{2} kT \qquad \Delta\bar{E} = \frac{3}{2} k (T_2 - T_1) \qquad 3{,}68 \cdot 10^{-18} = \frac{3}{2} k \cdot 100 \qquad k = \frac{2 \cdot 3{,}68 \cdot 10^{-18}}{100 \cdot K} \quad 1{,}38 \cdot 10^{-16}$$

Řeš.

(1) za 15.

za 1 částici $H = \frac{p^2}{2m} + ax^2 + bx$

$$Z = \frac{1}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dp\; e^{-\beta\left(\frac{p^2}{2m} + ax^2 + bx\right)}$$

$$ax^2 + bx = a\left[x^2 + \frac{b}{a}x\right] = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right]$$

$$x + \frac{b}{2a} = x' \qquad dx = dx'$$

$$Z = \frac{1}{h}\, e^{\beta\frac{b^2}{4a}} \int_{-\infty}^{\infty} dp\, e^{-\beta\frac{p^2}{2m}} \cdot \int_{-\infty}^{\infty} dx'\, e^{-\beta a x'^2}$$

$$t^2 = \frac{\beta}{2m}p^2 \qquad u^2 = a\beta x'^2$$

$$dp = \sqrt{\frac{2m}{\beta}}\,dt \qquad dx' = \frac{1}{\sqrt{a\beta}}\,du$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(t^2+u^2)}\,dt\,du$$

pa rovn. polár. souřad.

$$Z = \frac{1}{h}\, e^{\frac{\beta b^2}{4a}} \cdot \left(\frac{2m}{\beta}\right)^{1/2} \cdot \left(\frac{1}{a\beta}\right)^{1/2} \underbrace{\int_{-\infty}^{\infty} e^{-t^2}dt}_{2\cdot\frac{\sqrt{\pi}}{2}} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-u^2}du}_{\sqrt{\pi}}$$

$$\underline{Z = \frac{\pi}{h}\, e^{\frac{\beta b^2}{4a}} \cdot \frac{1}{\beta}\sqrt{\frac{2m}{a}}}$$

$$F = -\frac{1}{\beta}\ln Z = -kT\left[\ln\frac{\pi}{\beta h}\sqrt{\frac{2m}{a}} + \frac{b^2}{4a}\cdot\frac{1}{kT}\right]$$

$$U = -\frac{\partial}{\partial\beta}\ln Z = -\frac{\partial}{\partial\beta}\ln\frac{1}{\beta}e^{-\frac{b^2}{4a}\beta} = \frac{1}{\beta} + \frac{b^2}{4a}\beta$$

$$\underline{U = kT + \frac{b^2}{2a}\cdot\frac{1}{kT}}$$

$$H = \underbrace{\frac{p^2}{2m} + ax^2}_{\text{harm. osc.}} + bx$$

$$C_V = k - \frac{b^2}{2a}\cdot\frac{1}{kT^2}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V$$

II z.

② Naći omjer broja atoma He u sedimentnoj ravnoteži na temp. od 300 K u stupcu visokom 1 km i beskon. visokom stupcu

③ Sistem N-atoma, ∀ energije $E_1 = 2\,eV$ ili $E_2 = 5\,eV$

Izrač. temp. sistema, ako je broj molek. s E_1

$$N_1 = \frac{1}{3}N \qquad N_2 = \frac{2}{3}N$$

$$P_r = \frac{e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} \qquad \bar{E} = \frac{3}{2}kT \qquad \bar{E} = \frac{N_1 E_1 e^{-\beta E_1} + N_2 E_2 e^{-\beta E_2}}{N_1 e^{-\beta E_1} + N_2 e^{-\beta E_2}}$$

$$\bar{E} = \frac{\frac{1}{3}\cancel{N}E_1 + \frac{2}{3}\cancel{N}E_2}{\cancel{N}} = \frac{3}{2}kT$$

① Neka se sistem sastoji iz N magn. momenata koji su svi usmjereni u istom smjeru i na temp. $T = 0°K$

$\uparrow\uparrow\uparrow$
$\uparrow\uparrow\uparrow$ N
$\uparrow\uparrow\uparrow$

Pretpostavimo da porastom temp. može doći do preokretanja momenata u suprotni smjer, pri čemu se mijenja energija sistema kao ovdje za

$\Delta E = 2n\mu$ (n - broj preokrenutih momenata)

Naći: U, F, S, C na $T \neq 0$ za $N \to \infty$

(μ - magn. moment)

$$Z = \int_\Gamma e^{-\beta H} d\phi = \left(\int_\Gamma e^{-\beta E} d\phi'\right)^N$$

$$Z = \int_{\substack{N=0\\N=1\\\vdots\\N=N}} d\phi \Rightarrow Z = \sum_{n=0}^{N} e^{-2n\mu\beta} = \sum_{n=0}^{N} \left(e^{-2\beta\mu}\right)^n =$$

$$P_r = \frac{e^{-\beta E_r}}{\sum_r e^{-\beta E_r}}$$

$$U = \bar{E}_r = \sum_r E_r \cdot P_r = \frac{\sum_r E_r e^{-\beta E_r}}{\sum_r e^{-\beta E_r}}$$

N $E_1 = N_1$, E_2 N_2 $P_1 = \frac{1}{3}$ $P_2 = \frac{2}{3}$

$$\bar{E} = \sum_{r=1}^{2} E_r \cdot P_r = \frac{1}{3}E_1 + \frac{2}{3}E_2$$

$$Z = \frac{1 - e^{-2N\mu\beta}}{1 - e^{-2\mu\beta}}$$

$$U = -\frac{\partial}{\partial\beta} \ln Z = -\frac{1 - e^{-2\mu\beta}}{1 - e^{-2N\mu\beta}} \cdot \frac{\partial}{\partial\beta}\left(\frac{1 - e^{-2N\mu\beta}}{1 - e^{-2\mu\beta}}\right)$$

$$= -\frac{1 - e^{-2\mu\beta}}{1 - e^{-2N\mu\beta}} \cdot \frac{+2N\mu e^{-2N\mu\beta}(1 - e^{-2\mu\beta}) - (1 - e^{-2N\mu\beta})\cdot(2\mu e^{-2\mu\beta})}{(1 - e^{-2\mu\beta})^2}$$

$$F = -kT \ln Z =$$

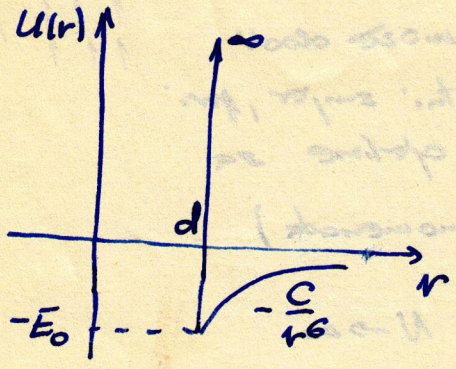
$$S = \frac{U - F}{T} \qquad C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

$N \to \infty$ $\quad U_\infty = 2\mu \dfrac{1}{e^{\frac{2\mu}{kT}} - 1} \qquad F_\infty = kT \ln\left(1 - e^{-\frac{2\mu}{kT}}\right)$

$$C_{V\infty} = k\left(\frac{2\mu}{kT}\right)^2 e^{\frac{2\mu}{kT}} \cdot \frac{1}{\left(e^{\frac{2\mu}{kT}} - 1\right)^2}$$

$T \to 0$ $\quad U_\infty \to 0$ $\quad S_\infty \to 0$ $\quad C_V \to 0$ $\qquad$ model feromagnetika

① Promatramo razrj. neidealni plin od N atoma. Pretpostavimo da između dva atoma djeluje kratkodosežni potencijal na sliv. :



Izračunati prvi virijalni koef. B(T) u razvoju u slučaju visokih temp.

$$P = \frac{NkT}{V}\left(1 + \frac{N}{V}B(T)\right)$$ razvoj po gustoći

$$Z = Z_0 \cdot Z_{int.}$$

$$B(T) = \frac{1}{2}\int\left(1 - e^{-\beta U(r)}\right)d^3r$$

$$F = F_0 + \frac{kTN^2}{V}B(T)$$

$$P = \frac{kTN}{V}\left(1 + \frac{N}{2}B(T)\right)$$

$$B(T) = \frac{1}{2}\int\left(1 - e^{-\beta U(r)}\right)d^3r = \frac{1}{2}\cdot 4\pi\left[\int_0^d\left(1 - e^{-\beta U(r)}\right)r^2dr + \int_d^\infty\left(1 - e^{-\beta U(r)}\right)r^2dr\right]$$

$$= \frac{2\pi d^3}{3} + 2\pi\int_d^\infty\left(1 - e^{-\beta U(r)}\right)r^2dr$$

$$b = \frac{2\pi d^3}{3} \qquad -a = 2\pi kT\int_d^\infty\left(e^{-\frac{U(r)}{kT}} - 1\right)r^2dr \quad \Rightarrow B(T) = b - \frac{a}{kT}$$

→ Integral nemožemo opć. riješiti, nego ga specificiramo za visok. temp. i razrjeđenja tako da nema interakcije → Tada $kT \gg E_0$

$$e^{-\frac{U(r)}{kT}} < e^{-\frac{E_0}{kT}} < 1$$

$$e^{-\frac{U(r)}{kT}} = 1 - \frac{U(r)}{kT} \quad \text{za } U(r) \ll$$

$$a = -2\pi \int_d^\infty U(r)\, r^2 dr = \frac{2\pi C}{d^3} \qquad E_0 = -\frac{C}{d^6}$$

$$\Rightarrow a = b E_0$$

$$B(T) = b - b\frac{E_0}{\varkappa T} > 0 \qquad \text{zbog } \varkappa T \gg E_0$$

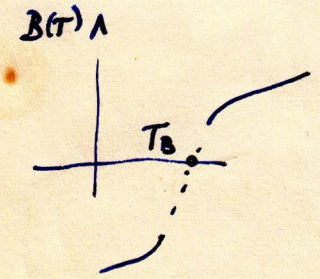
$$P = \frac{N\varkappa T}{V}\left(1 + \frac{N}{V} B(T)\right)$$

$$F = F_0 + \frac{\varkappa T N^2}{V} b\left(1 - \frac{E_0}{\varkappa T}\right)$$

$$U = U_0 - \frac{aN^2}{V} \qquad U_0 = \frac{3}{2} N\varkappa T$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = S_0 - \frac{\varkappa N^2 b}{V}$$

za $E_0 \gg \varkappa T$ $\quad \frac{E_0}{\varkappa T} \gg 1 \Rightarrow -\frac{U(r)}{\varkappa T} \gg 1$



$B(T) = 0$

zato u nekoj okolini T_B plin se ponaša kao idealan

T_B - Boylova temp.